



MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 16, September 5, 2016

1. Let $f(x) = (x + 1)(x + 2)(x + 3) \dots (x + n)$, and denote by S_o and S_e the sum odd and even coefficients of the polynomial $f(x)$, respectively. Then

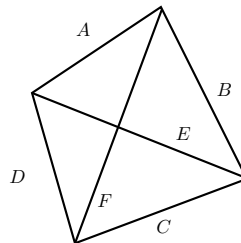
$$S_o + S_e = f(1) = 2 \times 3 \times \dots \times n \times (n + 1) = (n + 1)!,$$

and

$$S_e - S_o = f(-1) = 0.$$

Hence, $S_o = \frac{(n+1)!}{2}$.

2. Let A, B, C, D be the length of the edges, and E, F the length of the diagonals of the polynomial formed by the four points; as shown below.



It is given that

$$A + E + D = A + F + B = B + E + C = C + F + D. \tag{1}$$

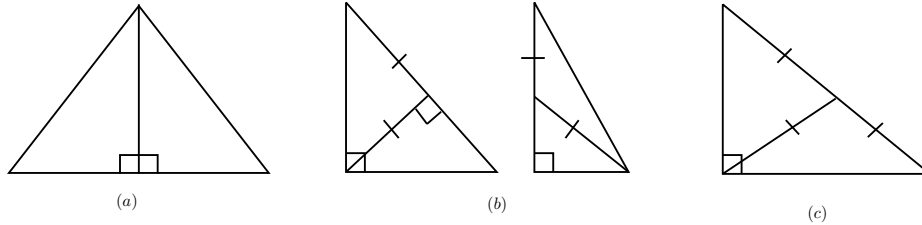
By the first and third terms of (1) we have $A + D = B + C$, and by the second and fourth terms of (1) we have $A + B = C + D$. Therefore $A = C$ and $B = D$. Furthermore, by the first and second terms of (1) we have $E + D = F + B$, and by third and fourth terms of (1) we have $B + E = F + D$. Therefore $E = F$. Thus, the polynomial formed must be a rectangle.

3. We show that

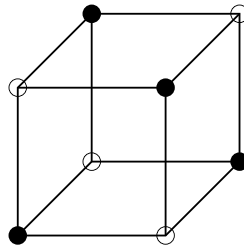
- (a) Any triangle can be dissected into 2 right-angled triangles.
- (b) Any right-angled triangle can be dissected into 1 right-angled triangle, and 1 isosceles triangle.

(c) Any right-angled triangle can be dissected into 2 isosceles triangles.

Then given any triangle, we can apply the above three dissecting operation, to form any $n \geq 4$ isosceles triangles we wish. The following diagrams illustrates the listed constructions



4. Place 4 black and 4 white dots on the corners of a cube, such that for any black dots the three adjacent corners are white dots, and visa versa for the white dots; as shown below.



If the black dots represent the number 1, and the white dots represents the number 0. Then at each step, all black dots turn into white dots, and all white dots turn into black dots. Hence, after ten steps, the positions of the black and white dots are the same as the initial configuration, but they are not all equal initially.

5. If we set the left turns as -90° rotations, then the right turns are 90° rotations. Since the car is not allow to pass through any place twice, in order for the car to return to its initial location, it must have rotated a net total of 360° or -360° . Hence it has made either 104 or 96 right turns.
6. We first show that the three hands coincide only at 12 : 00 or 24 : 00. Suppose this occurs again. Consider the angular distance θ covered by the hour hand where $0^\circ < \theta < 360^\circ$. The angular distance covered by the minute hand is $360^\circ n + \theta$, where n is the number of revolutions it has made. Since the minute hand moves at 12 times the speed of the hour hand, $360^\circ n + \theta = 12\theta$, so that $\theta = 360^\circ \frac{n}{11}$. The angular distance covered by the second hand is $360^\circ m + \theta$, where m is the number of revolutions it has made. Since the second hand moves at 720 times the speed of the hour hand, $360^\circ m + \theta = 720\theta$, so that $\theta = 360^\circ \frac{m}{719}$. From $\frac{n}{11} = \frac{m}{719}$, n must be a multiple of 11 and m a multiple of 719 as 11 and 719 are relatively prime. However, this contradicts $0^\circ < \theta < 360^\circ$. This justifies the Steve's claim. If there are two indistinguishable times within a twelve-hour period, shift the times so that one of them is at 12 : 00 or 24 : 00 and the other not. However, since one set of hands coincide, so must the other, and we have already proved that this is not possible

Senior Questions

1. By similar triangles it is easy to see that the typical point on the long diagonal of the cube has coordinates $(x, y, z) = (20t, 18t, 15t)$ with $0 \leq t \leq 1$. We can evaluate the number of unit cubes the diagonal line from $(0, 0, 0)$ to $(20, 18, 15)$ passes through, by considering the number of times the line intersects with the xy , xz and yz faces of each unit cube.

Let A, B, C represent the event of the line passing through the xy , xz , yz face of any unit cube, respectively. Then the number of times the line passes through the xy face of the unit cubes is $|A| = 15$, because this happens exactly when the z -coordinate of $(20t, 18t, 15t)$ is an integer, which occurs 15 times for $0 \leq t < 1$. Similarly, $|B| = 18$ and $|C| = 20$.

However, we can not just add $|A|$, $|B|$ and $|C|$ to obtain the number of unit cubes the long diagonal passes through, because we would double counted the event of this line passing through the edge of the unit cubes; i.e one such event is $A \cap B$ which represents the line passing through the edge of the cubes parallel to the x -axis. We have $|A \cap B| = 3$, since $A \cap B$ occurs when both the y and z coordinates of $(20t, 18t, 15t)$ is an integer for $0 \leq t < 1$; that is, the number of common factors in 18 and 15. Similarly, $|A \cap C| = 2$ and $|B \cap C| = 5$.

Finally, we must also take into consideration of the corner of the unit cubes; $|A \cap B \cap C| = 1$. Therefore, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 44.$$

The above formula is the well-know inclusion-exclusion principle https://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle.

2. A collection of 2016 numbers consists of one zero and 2015 ones.
 - (a) No, the average of the 2016 numbers does not change under the given operation. Hence, if all the numbers are equal, then they must all equal to the average of the initial set, which is $\frac{2015}{2016}$. Note that this process can only generate numbers of the form $\frac{n}{2^a}$. for some integers a and n . Since $2^{10} < 2016 < 2^{11}$, we can not possibly obtain $\frac{2015}{2016}$.
 - (b) Yes
3. Let the parabolas be $y_i = x^2 + a_i x + b_i$, $1 \leq i \leq 4$. Now y_i and y_j intersect if and only if $a_i \neq a_j$ (otherwise parabolas y_i and y_j are identical). Moreover, y_i and y_j intersect at exactly one point with $x = \frac{b_i - b_j}{a_j - a_i}$. Since we have only four points of intersection, we must have two distinct values of a_i , each appearing twice. Hence we may assume that $a_2 = a_1$ and $a_4 = a_3$. By symmetry, we may assume that $b_1 < b_2$, $b_3 < b_4$ and $a_1 < a_3$. This means that y_1 is below y_2 , y_3 is below y_4 and the common axis of y_1 and y_2 is to the right of the common axis of y_3 and y_4 . It follows that the rightmost point of intersection is that of y_2 with y_3 while the leftmost point of intersection is that of y_1 with y_4 . The sum of their x-coordinates is

$$\frac{b_1 - b_4}{a_3 - a_1} + \frac{b_2 - b_3}{a_3 - a_1} = \frac{b_1 + b_2 - b_3 - b_4}{a_3 - a_1}.$$

The sum of the x-coordinates of the other two points of intersections is

$$\frac{b_1 - b_3}{a_3 - a_1} + \frac{b_2 - b_4}{a_3 - a_1} = \frac{b_1 + b_2 - b_3 - b_4}{a_3 - a_1},$$

as well.