MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 2, May 7, 2016

1. If the first digit of n is $1, 2$ or 9, then there is nothing to prove. If the first digit of n is 3, then the first digit of $3n$ is 9. If the first digit of n is $4, 5$ or 6, then the first digit of $3n$ is 1. If the first digit of n is 7 or 8, then the first digit of $3n$ is 2. This completes the proof, has we have exhausted all possibilities.

2. How many numbers between 100 and 500 that are divisible by 7 but not by 21.

   The first and last numbers divisible by 7 between 100 and 500 are 105 and 497 respectively. Therefore, there are $(497 - 105) \div 7 = 56$ numbers between 100 and 500 that are divisible by 7. Since $21 = 3 \times 7$, the round-down of $1/3$ of the 56 numbers that are divisible by 7 is also divisible by 21. Therefore, the answer is the rounded-up of $56 \times 2/3 = 37.222\ldots$, which is 38.

3. Let ABC and DEF be right-angled triangles, with AF and DC their respective altitudes; see figure below. Point G is the intersection of AC and DF. Point H is such that GH is perpendicular to BC. Given $AF = 6$, $GH = 4$ and $FC = 9$.

   (a) Since ABC is a right-angled triangle, by the Geometric mean theorem (see `https://en.wikipedia.org/wiki/Geometric_mean_theorem`) one has $|AF|^2 = |BF| \times |FC|$, where the notation $|AF|$ here means the length side AF. Therefore, $|BF| = 6^2/9 = 4$; $|BC| = 13$.

   (b) Note that the triangles ABC, DEF and GFC are similar. Therefore,

   $$\frac{|FH| + |HC|}{|DC|} = \frac{|FH|}{|GH|} \quad \text{and} \quad \frac{|FH| + |HC|}{|AF|} = \frac{|HC|}{|GH|}.$$

   Solving the above system of equations gives $|DC| = 12$. Hence, the area of $AGDEB$ is $\frac{1}{2}(|AF| \times |BC| + |DC| \times |FE| - |GF| \times |HC|) = 171$.

4. Using the method of partial fractions, and then noting that the consecrative terms of

the series cancels,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2} = \sum_{i=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+2}$$
$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots$$
$$= \frac{1}{2} + \lim_{n \to \infty} \frac{1}{n}$$
$$= \frac{1}{2}$$

5. With out loss of generality, we can assume that $a^2 + b > a + b^2$. Let $p := a^2 + b - a - b^2$ be a prime number. We can write p as

$$p = a^2 + b - a - b^2$$
$$= a^2 - b^2 + b - 1$$
$$= (a - b)(a + b) - (a + b)$$
$$= (a - b)(a + b - 1).$$

Since p is prime, p has no divisors. Hence, by the last line of the above displayed equation, either $a - b = 1$ or $a + b - 1 = 1$. The former is impossible since $a - b < a + b - 1$. Therefore, we must have $a - b = 1$ and $a + b - 1 = p$. Hence, $2a - 1 = 1 + p$ so that $a = p/2 + 1$. But p is prime and a an integer, this can only happen when $p = 2$; $a = 2$ and $b = 1$ is the only solution.

6. Suppose Ben does his calculations in base b. Then in the usual base 10 system, the number 11 represents $b + 1$ and the number 52 represente $5b + 2$. Hence, the equation on the first line reads:
$$m^2 - (b + 1)m + 5b + 2 = 0.$$

Since $m = 7$ is a solution, we deduce that $49 + (b + 1) \times 7 + 5b + 2 = 0$, which implies $b = 22$. Therefore, the equation on the first line in the based 10 system is:
$$m^2 - 23m + 122 = 0.$$

Therefore, the other solutions is $m = 16$. It is easy to work out what is on the second line now because we know that Ben is working in a based 22 system.

Senior Questions

1. Note that
$$(f(x) - (\lambda_1 + \lambda_2 x))^2 \geq 0,$$

Thus

$$\int_0^1 (f(x) - (\lambda_1 + \lambda_2 x))^2 dx$$
$$= \int_0^1 (f(x))^2 dx - \left(2\lambda_1 \int_0^1 f(x)dx + 2\lambda_2 \int_0^1 xf(x)dx\right) + \int_0^1 (\lambda_1 + \lambda_2 x)^2 dx$$
$$= \int_0^1 (f(x))^2 dx - \left(2\lambda_1 + 2\lambda_2 + \lambda_1^2\right) + \lambda_1\lambda_2 + \lambda_2^2/3 \geq 0.$$

2

Therefore,

$$\int_0^1 (f(x))^2 dx \geq 2\lambda_1 + 2\lambda_2 - \left(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2/3\right). \tag{1}$$

In particular, the minimal of $\int_0^1 (f(x))^2 dx$ is attained when the RHS of (1) is minimal. Let $F(\lambda_1, \lambda_2) = 2\lambda_1 + 2\lambda_2 - \left(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2/3\right)$. To find the critical points of F, we solve the system

$$\frac{\partial F}{\partial \lambda_1} = -2 + 2\lambda_2 + \lambda_2 = 0, \tag{2}$$

and

$$\frac{\partial F}{\partial \lambda_2} = -2 + \lambda_1 + \frac{2\lambda_2}{3} = 0. \tag{3}$$

which gives $\lambda_1 = -2$ and $\lambda_2 = 6$. It is easy to check this critical point is minimal using the second partial derivative of F. It remains to show that these values of λ_1, λ_2 are consistent with the constraint $\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1$. To check this, we solve the system

$$\int_0^1 f(x)dx = \int_0^1 (\lambda_1 + \lambda_2 x)dx = \lambda_1 + \lambda_2/2 = 1,$$

and

$$\int_0^1 xf(x)dx = \int_0^1 (\lambda_1 x + \lambda_2 x^2)dx = \lambda_1/2 + \lambda_2/3 = 1.$$

The solution of the above system is $\lambda_1 = -2$ and $\lambda_2 = 6$.

2. Flatten the cube as shown below. We will use the symmetry of the objects to come up with a winning strategy for Jane. Note that the flattened cube is symmetric in all directions with respect to the center red object. Thus, if Jane takes out the red cube with her first move, then whichever object(s) John takes out, Jane can copy him symmetrically. For example, if John takes out the object at the most-top, left corner, then Jane will take out the object at most-bottom, right corner. This strategy will ensure that Jane always wins.
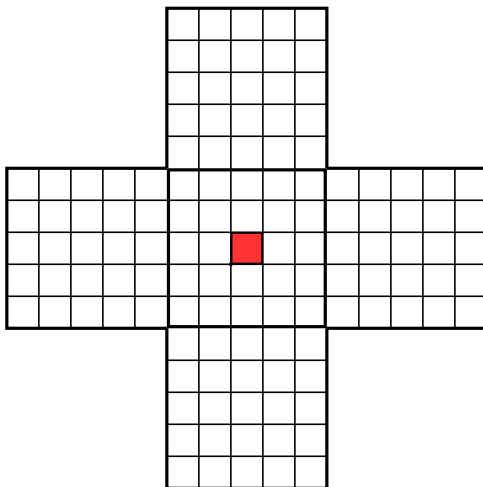
3. If $a = 0$, then $b = 2$ is the only solution.

   Now if $a > 0$, then 3×2^a is always even, so that b is always odd. Hence $b = 2k + 1$ for some integer k. Hence, $3 \times 2^a = 4k^2 + 4k = 4(k+1)k$.

   If $a = 1$, then $3 = 2(k+1)k$, which has no solutions.

   If $a = 2$, then $3 = (k+1)k$, again has no solutions.

   If $a > 2$, then $3 \times 2^{a-2} = (k+1)k$. Since 2 and 3 are coprime, if k is odd, then $k + 1 = 2^{a-2}$ and $k = 3$. Else if k is even, then $k + 1 = 3$ and $k = 2^{a-2}$. Hence, we conclude that $b = 7$, $a = 4$ and $b = 5$, $a = 3$ are the solutions for the $a > 2$ case.