MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 13, 2016

1. Let X be the total age of the committee members 4 years ago. Since there are 10 committee members, the total age of the members will increase by 4×10 in 4 years time; that is, the total age of the committee members currently is $X + 40$. Now, since only one member is replaced, the new member must be 40 years younger than the member being replaced.

2. Expanding and simplifying the expression $(1 + k)^3 - k^3$, we have

$$\begin{aligned}(1 + k)^3 - k^3 &= (1 + 3k + 3k^2 + k^3) - k^3 \\ &= 1 + 3k + 3k^2.\end{aligned}$$

Hence,

$$\begin{aligned}3k^2 &= (1 + k)^3 - k^3 - 3k - 1 \\ k^2 &= \frac{1}{3}[(1 + k)^3 - k^3 - 3k - 1]. \qquad (1)\end{aligned}$$

Therefore, we need to sum the RHS of (1) from $k = 1$ to $k = n$. Consider the sum of $[(1 + k)^3 - k^3]$ from $k = 1$ to $k = n$, one has

$$\begin{aligned}&(2^3 - 1^3) + (3^3 - 2^3) + \ldots + [n^3 - (n - 1)^3] + [(n + 1)^3 - n^3] \\ =& - 1^3 + (2^3 - 2^3) + (3^3 - 3^3) + \ldots + [(n - 1)^3 - (n - 1)^3] + [(n)^3 - (n)^3] + (n + 1)^3 \\ =&(n + 1)^3 - 1.\end{aligned}$$

Which implies

$$\begin{aligned}[(n + 1)^3 - 1] - 3(1 + 2 + \ldots + n) - n &= [(n + 1)^3 - 1] - 3\frac{n(n + 1)}{2} - n \\ &= (n + 1)[(n + 1)^2 - 3\frac{n}{2} - 1] \\ &= n(n + 1)(n + 1/2). \qquad (2)\end{aligned}$$

Therefore, by (1) and (2)

$$1^2 + 2^2 + 3^2 + \ldots + n^3 = \frac{n(n + 1)(2n + 1)}{6}.$$

3. We reduce 21 to prime factors, $21 = 3 \times 7$. Since $7 > 3$ the problem reduces to finding the higher power of 7 in 25!. The only factors of 7 come from $7, 14, 21$, which have one factor of 7 each. Therefore, the greatest power of 21 which divides 25! is 3.

4. Rectangle.

5. By the definition of perfect squares, we want to find integers n such that

$$n^2 - 3n - 126 = K^2$$

where $K \geq 0$ is an integer. We subtract K^2 from both sides and use the quadratic equation. This gives us the solutions

$$n = \frac{3 \pm \sqrt{513 + 4K^2}}{2}. \tag{3}$$

Since n is an integer, $\sqrt{513 + 4K^2}$ must be an odd integer; that is, $513 + 4K^2 = M^2$, where $M > 0$ is an odd integer. Rearranging and applying the difference of 2 squares, we get

$$(M + 2K)(M - 2K) = 513. \tag{4}$$

To solve the above (4), we note that $(M + 2K)$ and $(M - 2K)$ are integers, and $513 = 3^3 \times 19$. Therefore, to solve for K we can try every possible pair of integers, each containing factors of only 3 or 19, such that their product is 513. For example, for the integer combination $3, 171$ gives us the simultaneous equation

$$M + 2K = 171, \qquad M - 2K = 3$$

which has solution $M = 87, K = 42$. Substituting this into equation (3) gives $n = -42, 45$. For the integer combination 9×57, we get $M = 33, K = 12$ and $n = -15, 18$. For the combination 19×27 we get $M = 23, K = 2$ and $n = -10, 13$. Finally the integer combination 1×513 gives us $M = 257, K = 128$ and $n = -127, 130$. So finally, we have solutions $n = -127, -42, -15, -10, 13, 18, 45, 130$.

6. (a) Consider a $m \times n$ rectangle where $1 \leq m \leq 6$ and $1 \leq n \leq 7$ and m, n integers. It is clear that we can fit $(6 - m + 1)(7 - n + 1)$ of the $m \times n$ rectangles in a 6×7 rectangle. To find the total number of rectangles of integer side lengths, we need to sum over all m, n. This is given by $(1+2+\ldots+6)(1+2+\ldots+7) = 21*28 = 588$.

   (b) Consider a $m \times m$ square where $1 \leq m \leq 6$ and m integer. We can fit $(6 - m + 1)(7 - m + 1)$ of the $m \times m$ square into a 6×7 rectangle. Summing for $m = 1, \ldots, 6$ we get $1 * 2 + 2 * 3 + \ldots + 6 * 7 = 112$.

Senior Questions

1. We can manipulate the summation as follows

$$\begin{aligned}
\sum_{k=1}^{28} a_k &= \sum_{k=4}^{30} a_k + a_1 + a_2 + a_3 - a_{29} - a_{30} \\
&= \sum_{k=4}^{30} a_{k-1} + a_{k-1} + a_{k-3} + a_1 + a_2 + a_3 - a_{29} - a_{30} \\
&= \sum_{k=3}^{29} a_k + \sum_{k=2}^{28} a_k + \sum_{k=1}^{27} a_k + a_1 + a_2 + a_3 - a_{29} - a_{30} \\
&= 3\sum_{k=1}^{28} a_k - a_1 - a_2 + a_{29} - a_1 - a_{28} + a_1 + a_2 + a_3 - a_{29} - a_{30} \\
&= 3\sum_{k=1}^{28} a_k - a_{28} - a_{30},
\end{aligned}$$

where the last equality is due to the fact that $a_1 = a_2 = a_3$. Therefore, by rearrange the above displayed equation

$$\sum_{k=1}^{28} a_k = \frac{1}{2}(a_{30} + a_{28}) = 13346839.$$

Note that there is an error in the question: $a_{28} > a_{29} < a_{30}$ which doesn't really make sense for this sequence!

2. First we solve for x. For $n > 10$, the number $n!$ contain the factors $2, 5$ and 10, and therefore always has 00 as its last 2 digits. Hence, the last 2 digits of $\sum_{n=0}^{100} n!$ are equal to the last 2 digits of $\sum_{n=0}^{10} n!$, which is 14. Therefore, $x = 14$.

   Now, to find the last 2 digits of $\sum_{m=0}^{100} m^{14}$, is the same as solving for $\sum_{m=0}^{100} m^{14} \mod 100$ (see `https://en.wikipedia.org/wiki/Modular_arithmetic`). Let $m = (10b + r)$. We apply binomial theorem to $(10b + r)^{14}$, and note that first 12 terms of the binomial expansion is equal to $0 \mod 100$, so that

$$\begin{aligned}
\sum_{m=0}^{100} m^{14} &= \sum_{b,r=0}^{9} (10b + r)^{14} \mod 100 \\
&= \sum_{b,r=0}^{9} 140br^{13} + r^{14} \mod 100 \\
&= \sum_{r=0}^{9} 45 \times 140r^{13} + 10 \times r^{14} \mod 100 \\
&= \sum_{r=0}^{9} 10r^{14} \mod 100.
\end{aligned}$$

   Thus, the last digit of $\sum_{m=0}^{100} m^{14}$ is always 0. Therefore, we just need to find the second last digit of $\sum_{m=0}^{100} m^{14}$, which is equal to the last digit of $\sum_{m=0}^{9} r^{14}$. To find the last digit o $\sum_{m=0}^{9} r^{14}$, we note that for any integer k, one has $k^5 = k \mod 10$, thus $k^{13} = (k^5)^2 \times k^3 = k^5 = k \mod 10$, so that $k^{14} = k^2 \mod 10$. Therefore

$$\sum_{r=0}^{9} = r^{14} = \frac{9 \times 10 \times 119}{6} = 285 = 5 \mod 10,$$

where we have used the results of $Q2$ to obtain the second equality. Hence, the last two digits of $\sum_{m=0}^{9} r^{14}$ is 50.

3. The number of towns is finite and there are only six ways of entering and leaving any given town. Hence the traveller must eventually pass through some town T twice entering and leaving in the same direction. From this point on the traveller loops forever.

   Suppose that after two loops the traveller turns around and retraces his steps turning left where he turned right and vice versa. This reverse the journey staying on the loop hence the initial town is on the loop at leat once and up to six times.