

**MATHEMATICS ENRICHMENT CLUB.**  
**Problem Sheet 9, July 24, 2017**

1. What is the least positive integer  $n$  such that  $90 \times n$  is a cube?
2. Show that any straight line passing through the centre of a parallelogram (i.e. the intersection of the diagonals) divides the parallelogram into two equal areas.
3. A mathematics test has 5 questions on each of which people can score 0,1,2 or 3 marks. How many ways can a student receive a total of 12 marks for the test?
4. Use the fact that  $2xy = (x + y)^2 - x^2 - y^2$  to show that

$$2(b - c)(c - a) + 2(c - a)(a - b) + 2(a - b)(b - c) \leq 0$$

for all real numbers  $a, b, c$ .

5. Take any triangle  $ABC$  and show how to construct an equilateral triangle inside  $ABC$  whose vertices touch the sides of  $ABC$ . (Hint: Start by constructing an equilateral triangle outside  $ABC$  with  $AB$  as one of its sides.)
6. Imagine that we have a finite set  $A$  of integer numbers, that is, a collection of integers without repetition. Consider the set  $A + A$  of all possible sums of two numbers in  $A$ :

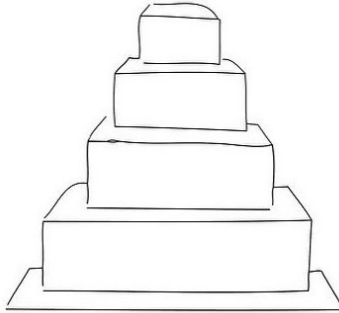
$$A + A = \{n : n = a + a' \text{ for some numbers } a, a' \text{ in } A\}.$$

We denote how many numbers there are in the set  $A$  by  $|A|$ .

- (a) Show that  $|A + A| \geq 2|A| - 1$ .
- (b) Show that if  $|A + A| = 2|A| - 1$ , then  $A$  is an arithmetic progression.

## Senior Questions

1. Imagine that you have a square based cake, like the one in the picture.



- (a) How would you cut it into 5 pieces of equal volume? How about 7 pieces?
  - (b) How about  $n$  pieces of equal volume?
2. Show that  $\log_2 3$  is not a rational number.