MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 12[1], August 14, 2017

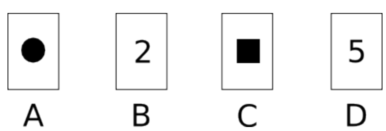
1. Let m and n be positive numbers satisfying:

   (a) The sums of its divisors coincide.

   (b) The sums of the reciprocal of its divisor also coincide.

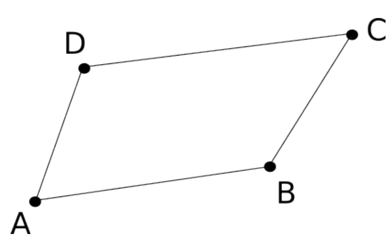
   Show that necessarily $m = n$.

2. Four cards are constructed so that there is either a circle or a square on one side and an odd or even number on the other side. The cards are placed on the other side. The cards are placed on a table as shown. Which cards must be turned to prove the following "Every square has an even number on the other side"?



3. A mathematician and her partner had dinner with four other couples. The guests shook hands but, of course, no one shook hands with themselves or with their partner. The mathematician noticed that the the mathematicians partner and the eight other guests each shook hands with a different number of people. With how many people did the mathematicians partner shake hands?

4. Let $ABCD$ be a convex quadrilateral (all interior angles are less than 180^{o} and the two diagonals both lie inside the quadrilateral).



   Show that the quadrilateral obtained by joining the midpoints of AB, BC, CD and DA has area exactly half the area of $ABCD$.

[1] Some of the problems were suggested by I. Shparlinski and T. Britz.

5. There are many examples of twin primes: pairs $(p, p+2)$ in which both are prime numbers (for example $(3, 5)$, $(11, 13)$, $(41, 43)$,...). In fact, mathematicians believe that there are infinitely many twin primes but we still do not know how to prove this.

   We will look at a much simpler case: let us define prime triplets as 3 consecutive odd primes $(p, p+2, p+4)$. Are there any besides $(3,\ 5,\ 7)$? If not, how do you prove it?

Senior Questions

We will devote these questions to understand a bit more in detail the Harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

1. Show that the Harmonic series is greater than any given number $M > 0$. Then show that if we only take the sum of reciprocals of even numbers, the sum is also greater than any given number.

2. If $n > 1$. Prove that the partial sum

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

   is never an integer.

3. Consider the sum of reciprocals of all natural numbers that do not contain the number nine (when written in decimal expansion)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + ...$$

   (we avoid $1/9,\ 1/19,\ 1/29, ...$ and so on).

   Show that in fact the sum of the remainder terms is smaller than 10.