MATHEMATICS ENRICHMENT CLUB.
Solution Sheet ?, July 24, 2017

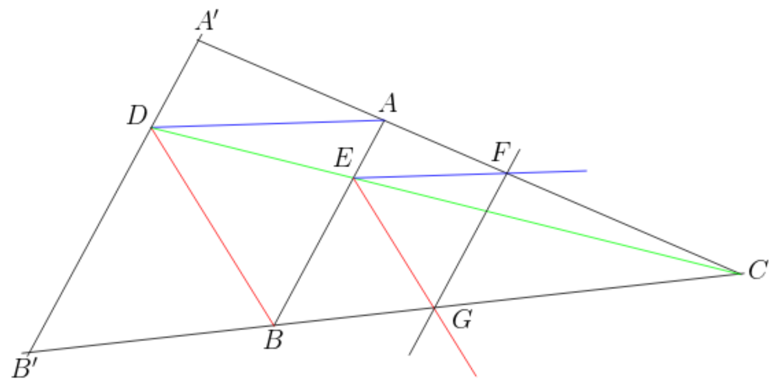
1. Note that $90 \times n = 2 \times 3^2 \times 5 \times n$, so if the product is a cube in particular must have 2^3, 3^3 and 5^3 as divisors. Thus, it suffices to choose $n = 3 \times 2^2 \times 5^2 = 300$, so we have
$$90 \times n = 2^3 \times 3^3 \times 5^3 = (2 \times 3 \times 5)^3.$$

2. The result is clear if the line drawn is either of the diagonals, for they are lines of symmetry. Suppose now that the line drawn is not a diagonal. With the diagonals also drawn, the parallelogram is divided into 6 triangles. Well leave the rest up to you, but these 6 triangles come in 3 pairs of congruent triangles, each having one of the pair on either side of the line, meaning that the line divides the parallelogram into two equal areas.

3. If we expand $(1 + x + x^2 + x^3)^5$, the coefficient of x^k will be precisely the number of ways that we can write k as a sum of 5 numbers in $\{0, 1, 2, 3\}$ (this follows from the simple fact that $x^a \cdot x^b = x^{a+b}$ and the coefficient of x_k will be the sum of the number of ways we get x^k in the expansion). Thus, it suffices to expand
$$(1 + x + x^2 + x^3)^5 = x^{15} + 5x^{14} + 15x^{13} + 35x^{12} + 65x^{11} + \cdots$$
so the number of ways to get 12 points in the test is precisely 35.

4. Note that it follows from the formula that, for any real a, b, c,
$$\begin{aligned} 2(b-c)(c-a) &= (b-a)^2 - (b-c)^2 - (c-a)^2 \\ 2(c-a)(a-b) &= (c-b)^2 - (c-a)^2 - (a-b)^2 \\ 2(a-b)(b-c) &= (a-c)^2 - (a-b)^2 - (b-c)^2 \end{aligned}$$
Now we combine these equalities, together with the observation that for any real number x we have $(-x)^2 = (-1)^2x^2 = x^2 \geq 0$, to get
$$2(b-c)(c-a) + 2(c-a)(a-b) + 2(a-b)(b-c) = -(b-a)^2 - (c-b)^2 - (a-c)^2 \leq 0.$$

5. Begin by constructing the equilateral triangle ADB. Draw the line CD to intersect AB at E. Draw EG parallel to DB and EF parallel to DA. Connect F and G, then the triangle EFG is equilateral. To prove this is true, construct $B'A'$ parallel to BA and passing through D, and show that $BB'D$ and $AA'D$ are similar to GBE and FAE respectively.



6. Let us write the elements in A as $a_1, ..., a_k$, with

$$a_1 < a_2 < \cdots < a_{k-1} < a_k.$$

(a) Note that we can construct the chain

$$a_1 + a_1 < a_1 + a_2 < a_1 + a_3 < \cdots < a_1 + a_k < a_2 + a_k < \cdots < a_{k-1} + a_k < a_k + a_k,$$

of elements in $A + A$ that has $2k - 1 = 2|A| - 1$ distinct elements.

(b) If $|A+A| = 2|A|-1$ this implies that the elements in the chain are all the possible ones. Thus, we can construct the following chains

$$\begin{aligned} &a_1 + a_1 < a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < \cdots < a_1 + a_k \ < a_2 + a_k < \cdots \\ &a_1 + a_1 < a_2 + a_1 < a_2 + a_2 < a_2 + a_3 < \cdots < a_2 + a_{k-1} < a_2 + a_k < \cdots \end{aligned}$$

of length $2|A| - 1$. Since element by element, both chains must coincide, in particular we have that for every $i = 1, \ldots, k-1$

$$a_2 + a_i = a_1 + a_{i+1}$$

which implies in particular that

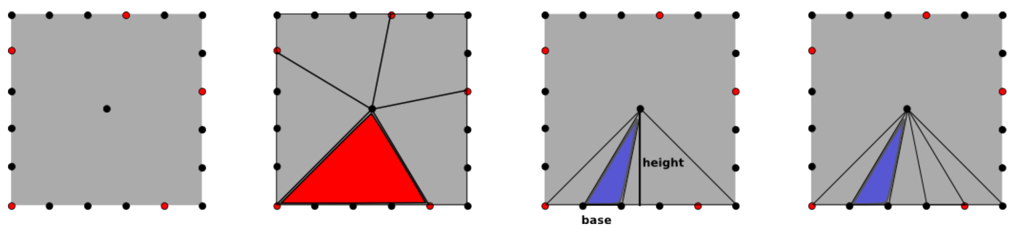
$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_k - a_{k-1} = d$$

or in other words: A is an arithmetic progression of difference d.

Senior Questions

1. We will resolve it for $n = 5$ and leave the generalisation to the reader. We will show how to cut a square into 5 pieces of the same area.

   To do so, we will divide each side of the square in 5 equal segments (dividing the whole perimeter in 20 equal parts, represented in the figure by black and red circles). Now, we will join the center of the square with every fourth vertex (coloured in red) and obtain 5 pieces with the same area.

   How can we be sure that every piece has the exact same area? Observe that if we connect the center with two consecutive vertices, the resulting triangle (like the blue one on the figure) will have the same area independently of the choice of vertices since the base and height are the same. To conclude note that every piece is formed by exactly 4 such triangles and therefore they all have the same area.

   Now, by cutting our cake following these lines we will be cutting the subsequent layers of our cake –which are all centered squares– forming pieces of equal volume on each layer.

2. Let us assume the contrary, that is

$$\log_2 3 = \frac{p}{q}, \qquad \text{for some coprime integers } p, q > 0.$$

   Then, in particular we have

$$2^p = 3^q.$$

   But the left hand side is an even number and the right hand side is odd, therefore such p, q do not exist.