



MATHEMATICS ENRICHMENT CLUB.
Solutions to Problem Sheet 16, September 11, 2017

1. We have

$$\begin{aligned} N! &= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 11 \times 7 \times 5^2 \times 3^4 \times 2^8 \\ &= (11 \times 7 \times 5^2 \times 3 \times 2^2) \times (3 \times 2^2)^3. \end{aligned}$$

So n^3 divides $N!$ if and only if n divides $3 \times 2^2 = 12$. The only options are:

$$n = 1, 2, 3, 4, 6, 12.$$

2. Adding the given equations and remembering the identity $\log X + \log Y = \log(XY)$ gives

$$\log(x^{2n+2}y^{2n+2}) = 2.$$

Using $\log(X^m) = m \log X$, we have

$$(2n + 2) \log(xy) = 2$$

and therefore

$$\log(x^n y^n) = n \log(xy) = \frac{2n}{2n + 2} < 1,$$

since $n \geq 1$.

3. Note that for any positive integer k we have

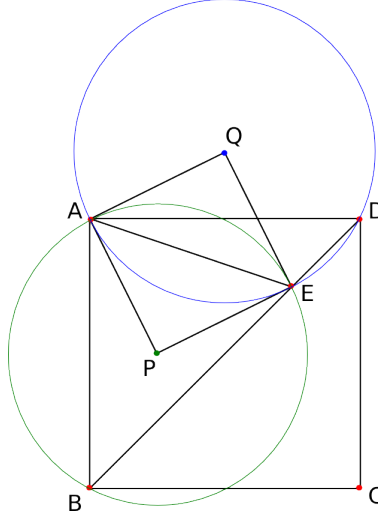
$$\frac{1}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k+1} - \sqrt{k}} = \frac{\sqrt{k+1} - \sqrt{k}}{k+1 - k} = \sqrt{k+1} - \sqrt{k}.$$

Thus we can rewrite our expression as

$$\begin{aligned} \frac{1}{\sqrt{2} + \sqrt{1}} + \cdots + \frac{1}{\sqrt{3457} + \sqrt{3456}} &= \\ &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{3456} - \sqrt{3455}) + (\sqrt{3457} - \sqrt{3456}) \\ &= \sqrt{3457} - 1 \end{aligned}$$

4. First we know that $\angle APE = 2\angle ABE = 90^\circ$ and $\angle AQE = 2\angle ADE = 90^\circ$.

Also, $\triangle APE$ is isosceles as $AP = PE$ (since equal radii.) and we have that $\angle PEA = \angle PAE = \frac{1}{2}\angle APE = 45^\circ$.



The same reasoning gives us that $\triangle AQE$ is isosceles as $QE = QA$ (equal radii.) which implies that $\angle QEA = \angle QAE = 45^\circ$.

All together implies that

$$\angle QAP = \angle QEP = 90^\circ.$$

Four right angles define a rectangle but we know $AP = PE$, so a rectangle with a pair of adjacent sides equal is a square.

5. Suppose that the number 2^n consists of a digit a followed by s further digits and 5^n consists of a followed by t further digits. Then we have

$$2^n = (a + x)10^s \quad \text{and} \quad 5^n = (a + y)10^t,$$

where $0 \leq x < 1$ and $0 \leq y < 1$. Multiplying these equations,

$$10^n = (a + x)(a + y)10^{s+t}$$

and so

$$(a + x)(a + y) = 10^{n-s-t}.$$

However, $n - s - t$ is an integer and $1 \leq (a + x)(a + y) < 100$ (since a is a digit in $1, 2, \dots, 9$), so either

$$(a + x)(a + y) = 1 \quad \text{or} \quad (a + x)(a + y) = 10.$$

In the first case we have $a = 1, x = 0$ and $y = 0$, so $2^n = 10^s$; since n is a positive integer, this is impossible.

In the second case we have

$$a^2 \leq (a + x)(a + y) < (a + 1)^2,$$

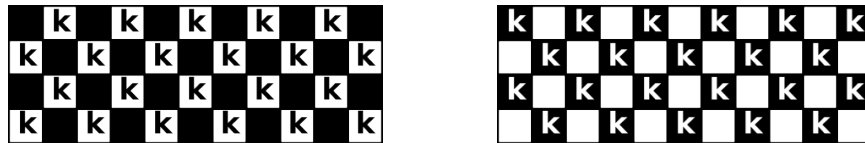
that is,

$$a^2 \leq 10 < (a + 1)^2,$$

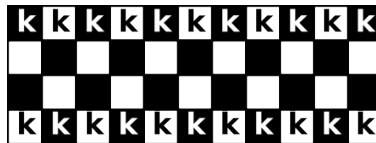
and so $a = 3$ is the only possible digit. And indeed, if $n = 5$ then $2^n = 32$ and $5^n = 3125$, both of which start with a 3.

Comment. In fact, it is possible to show that there are infinitely many such n : the numbers 2^n and 5^n both begin with 3 for $n = 5, 15, 78, 88, 98, 108, 118, \dots$

6. (a) A knight on a chessboard always moves to a square of a different colour. So if we put knights on the $2n$ white squares of a $4 \times n$ board, none will attack another. A second solution, of course, is to put the knights on the black squares.



A third solution is to place them on the two sides of length n : since there are two empty rows between them, the knights on one side cannot reach those on the opposite side.



- (b) Now suppose that there is a closed knights tour on the $4 \times n$ chessboard, and consider how we can use this tour to locate $2n$ nonattacking knights.

Since no two consecutive squares on the tour can be occupied by nonattacking knights, there are only two possible placements for the $2n$ knights: on every second square of the tour (odd positions), or on every other second square of the tour (even positions). But we know that there are in fact three ways to place the knights (perhaps more): the only possible conclusion is that the closed knights tour on the $4 \times n$ board cannot exist.

Senior Questions

1. Note that you're travelling on a straight line (i.e., distance=displacement), since your car cannot turn.

In that case, we can say that the position, given as a function $f(t)$ of time, has a derivative (instantaneous velocity) which is equivalent to $f'(t)$, which is the distance travelled from the starting position (the house). This means that $f'(t) = f(t)$.

It is well known that, in this case, $f(t) = Ce^t$ for some constant C . Taking into account that $f(0) = 1$ (we start at 1km from our house). Therefore, after one hour, we are e kms away.

2. The points of intersection are $A = (1, 0)$ and $B = (e, 1)$. In the region we are interested, $1 \leq x \leq e$, we have $0 < \log x < 1$ hence $0 < \log^2 x < \log x$. Which means that the graph of the function $\log x$ lies above the graph of $\log^2 x$.

The area we are looking for can be expressed as:

$$\int_1^e (\ln x - \ln^2 x) dx.$$

We have

$$\int_1^e \log x dx = [x \log x - x]_1^e = (1 \cdot e - e) - (0 - 1) = 1$$

and

$$\int_1^e \log^2 x dx = [x \ln^2 x - 2x \ln x + 2x]_1^e = e - 2.$$

Hence the area equals $3 - e$.

3. A simple proof by induction shows this property. The simplest case is a 2x2 chessboard. Clearly if one square is removed, the remainder can be tiled by one L-shaped tile. This is the base case.

Now, given a large chessboard of size $2n \times 2n$, with one square removed, it can be divided into four $2(n-1) \times 2(n-1)$ smaller chessboards (the corners of the original board). One of those corners contains the removed tile, so by the inductive step, it can be tiled by L-shaped tiles. Now notice that the remaining three $2(n-1) \times 2(n-1)$ boards all meet at a common corner at the center of the original board.

The 3 corner tiles there form an L-shaped set that can be covered by one tile. And by removing them, those 3 chessboards each have a tile removed, and by the inductive step they can be tiled by L-tiles as well. Thus we have covered the entire chessboard (apart from the removed square) by L-shape tiles. This proof actually leads to a method for constructing such a tiling, by successive applications of induction. As an example, consider Figure 1 above without any tiling and with one square removed (here, the black square). We divide the board into four 4×4 boards, one in each quadrant. The bottom left board can be tiled by the inductive step, because it has a square removed. And in the three other boards, we can cover the corner squares where they meet by a single L-shaped tiles (here, orange). These three 4×4 boards will then have a single tile removed, hence by the inductive step they can be tiled too.

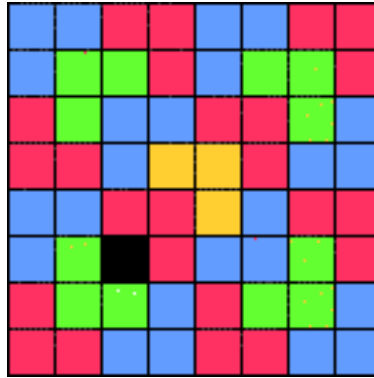


Figure 1: Example with $n = 3$

But how can each of these 4×4 boards with a square removed be tiled? We apply our inductive method again, cutting each into four 2×2 boards: one of which has a square removed, and the other three of which don't have a square removed. Cover their 3 center tiles by a (green) L-shaped tile. What remains uncovered are 2×2 boards with a square removed... these can easily be covered by (red and green) tiles.