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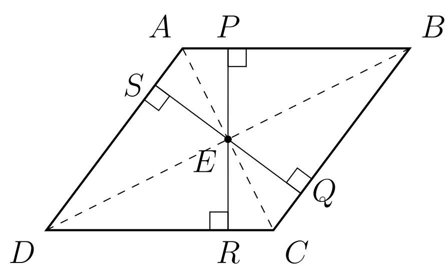
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 14, August 27, 2018

1. The sum of the first three cubes is $1^3 + 2^3 + 3^3 = 36$, which is divisible by 18. Find all triplets of consecutive natural numbers such that the sum of their cubes is divisible by 18.

2. Let ABC be a triangle, and let D be a point inside the triangle. Prove that the sum of segments connecting D to the three vertices is larger than the semi-perimeter of $\triangle ABC$.

3. The quadrilateral $ABCD$ is a rhombus. The two diagonals of $ABCD$ intersect at E; and P, Q R and S are the feet of perpendiculars from E to the sides of the rhombus.



   Prove that $PQRS$ is a rectangle [1].

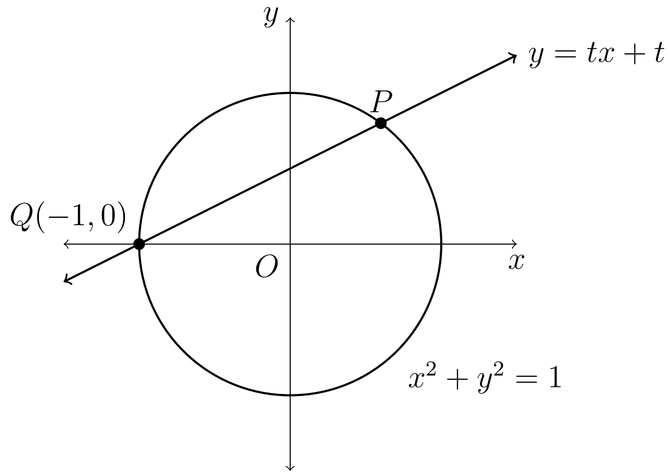
4. A thousand coins lie in one long row, all facing tails up. We first walk along the line and flip over every coin. Then we start again, flipping over every second coin, then again flipping over every third coin, then again flipping every fourth, and so on. After doing this a thousand times, so that the last time we flip only the last coin, how many coins are facing heads up?

5. Without using a calculator, determine which of $\sqrt[88]{88!}$ and $\sqrt[99]{99!}$ is larger.

[1] Questions (2) & (3) have been adapted from AP Kiselev *Kiselev's Geometry: Planimetry*, Tr. A Givental, 2006

Senior Questions

1. The diagram below shows the unit circle $x^2 + y^2 = 1$ and the line $y = tx + t$, where $t \in (0, 1)$. The line and the circle intersect at the points P and $Q(-1, 0)$.



   (a) Show that the coordinates of P are $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

   (b) The coordinates of P found in part (a) should remind you of the t-results from trigonometry. Explain what is going on here.

   (c) A Pythagorean triple is a set of three integers that could be the lengths of the sides of a right angled triangle; that is, three integers (a, b, c) such that $a^2 + b^2 = c^2$. The best known Pythagorean triple is $(3, 4, 5)$. We can use results from part (a) to generate Pythagorean triples. Explain how to do this, and show that there is an infinite number of right angled triangles with integer sides.

2. (a) Let $\mathcal{C}$ be a circle centred at O. Let P be a point lying in the plane. Show that the point on $\mathcal{C}$ lying closest to P is lies on the line joining O and P.

   (b) Now, let's prove essentially the same result using complex numbers.

   Let $z, w \in \mathbb{C}$ such that $z = r(\cos\theta + i\sin\theta)$ where r, a non-negative real number, and $\theta \in (-\pi, \pi]$ are fixed, and $w = \cos\phi + i\sin\phi$, where $\phi \in (-\pi, \pi]$. (That is to say, z is an arbitrary fixed point in the Argand plane and w lies anywhere on the unit circle.) Show that $|z - w|$ is minimised when $\phi = \theta$.