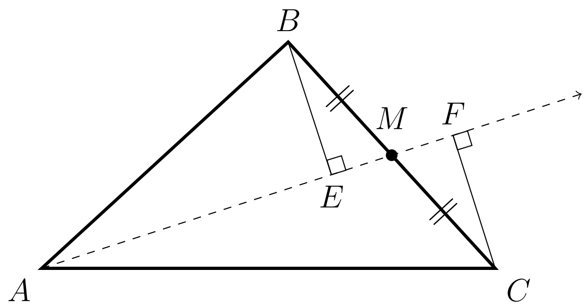
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 11, 13 August, 2018

1. Let BE and CF be perpendiculars dropped from B and C to AM, extended if necessary. We need to prove that $BE = CF$.



Since BE and CF are both perpendicular to AM, $\angle BED = \angle DFC = 90°$, and since AM is a median, $BM = CM$. Moreover, $\angle BDE = \angle CDF$, since $\angle BDE$ and $\angle CDF$ are vertically opposite. Thus $\triangle BDE \equiv \triangle CDF$ by AAS. Thus BE and CF are corresponding sides in congruent triangles and hence equal.

2. (a) To begin, $n^5 - 5n3 + 4n$ can be factored as $(n+2)(n+1)n(n-1)(n-2)$. That is, if n is an integer, then $n^5 - 5n3 + 4n$ is the product of five consecutive integers, and hence can be divided by each of 5, 4, 3 and 2. Thus it must be divisible by 120.

   (b) Conversely, suppose that 49 is a divisor of $n^2 + n + 2$ for some integer n. Then $n^2 + n + 2 = (n+4)^2 - 7(n+1)$ and both $(n+4)^2$ and $7(n+2)$ must be divisible by 49, or both $(n+4)$ and $(n+2)$ by 7. This is not possible.

3. If A was truthful about B coming second, then B must be lying about A coming second and C about B coming third, so the order would be ABC.

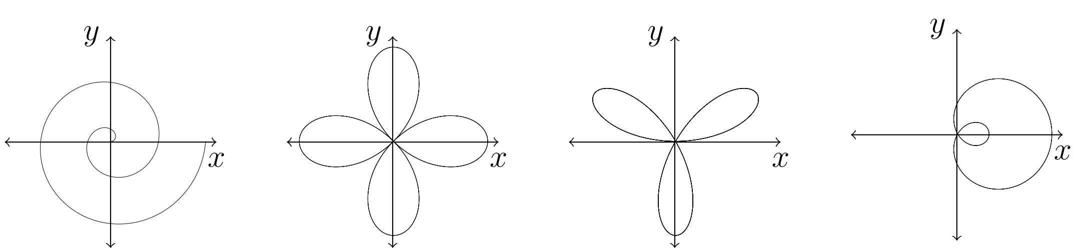
   If A was truthful about C coming first, then B must be lying that C was third and C about A coming first, so the order would be CAB. Either way, A beat B.

4. (a) $0.75_{10} = 0.11_2$, since $0.75 = \frac{1}{2} + \frac{1}{4} = 1 \times \frac{1}{2^1} + 1 \times \frac{1}{2^2}$
   (b) $0.96875_{10} = 0.11111_2$ in base 2.
   (c)
$$\frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^k} + \cdots = 0.\dot{1}_2 = 1$$
   If you are not convinced of this last fact, let $x = 0.\dot{1}_2$. Then
$$\begin{aligned} 2x &= 1.\dot{1}_2 && (1) \\ x &= 0.\dot{1}_2 && (2) \\ \hline x &= 1 && (1) - (2) \end{aligned}$$

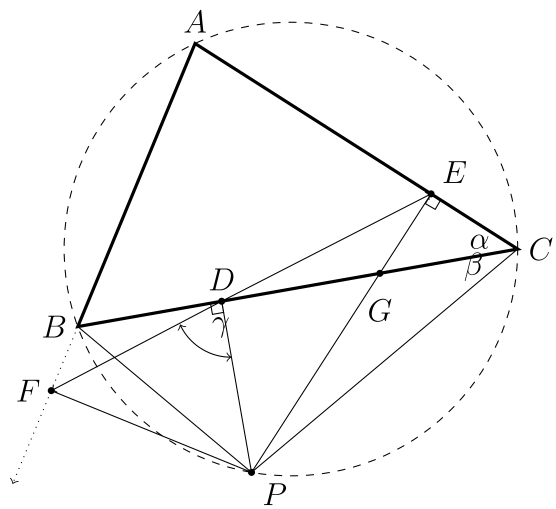
5. Firstly, we note that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. Then, we find the prime factorisation of $1729 = 7 \times 13 \times 19$. Thus the possible factors of 1729 are 1, 7, 13, 19, 91, 133, 247, and 1729 itself. If we assume that $x - y = 1$, then
$$x^2 + xy + y^2 = 1729.$$
   Furthermore, we can substitute $x = y + 1$ into this second equation, thereby obtaining a quadratic in y. In this case, the quadratic does not have integer solutions, as Δ is not a perfect square. However, continuing this way through all the possibilities, we obtain the solutions $(-1, 12)$, $(1, -12)$, $(-9, 10)$ and $(9, -10)$.

Senior Questions

1. (a) i. $(1, 1)$ ii. $(0, -1)$ iii. $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$. iv. $(-\sqrt{3}, 1)$
   (b) The graphs should be as follows:
   i. $r = \theta$ ii. $r = \cos(2\theta)$ iii. $r = \sin(3\theta)$ iv. $r = 1 + 2\cos\theta$



2. Join CP and PB as shown.



Let $\angle ACB = \alpha$, $\angle BCP = \beta$ and $\angle FDP = \gamma$. Let D and E be the feet of perpendiculars from P to the sides of the triangle as shown. Extend a line through D and E, and let F be the point of intersection of DE with the side AB (extended if necessary). We have to show that $\angle BFP = 90°$.

Now $\angle PDC = \angle PEC = 90°$, so $DECP$ is a cyclic quadrilateral. Thus $\angle DEP = \angle DCP = \beta$. Furthermore, by the angle sum of $\triangle ECG$, $\angle EGC = 90° - \alpha$. But $\angle EGC = \angle GDE + \angle DEG$, by the exterior angle theorem. And this implies that $\angle EDG = 90° - (\alpha + \beta)$. Since $\angle BDF$ and $\angle EDG$ are vertically opposite, $\angle BDF = \angle EDG = 90° - (\alpha + \beta)$. Consequently, $\gamma = \alpha + \beta$.

Since $BACP$ is a cyclic quadrilateral $\angle ABC + \angle ACP = 180°$. Thus $\angle ABP = 180° - \angle ACP = 180° - (\alpha + \beta)$, which implies that $\angle FBP = \alpha + \beta = \gamma$. Thus $\angle FBP = \angle FDP$, and so $FBDP$ is a cyclic quadrilateral also. Hence $\angle BFP + \angle BDP = 180°$, and so $\angle BFP = 90°$, as required.