



MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 13, 27 August, 2018

1. It is not possible. The sum of the digit sums of the numbers on the left-hand side of the equation is $1 + 2 + 3 + \dots + 9 = 45$, which is a multiple of nine, but 100 is not.
2. (a) We are choosing three objects from a possible 8, and order is not important, so there are ${}^8C_3 = 56$ triangles.
 (b) Most of the triangles in the cube are right-angled. However, we can create an equilateral triangle by joining together three diagonals of the faces of the cube, e.g. $\triangle AGE$. For each vertex of the cube, there is exactly one diagonal that can be chosen as the base to form an equilateral triangle. Thus there are eight acute triangles in total.
3. First note that we can buy exactly $10 = 5 + 5$ donuts. Now if we figure out the 9 ways of buying donuts so we get the smallest possible number ending with each digit, we can figure out larger orders by simply adding 10 to them.

1	$13 + 13 + 5 = 31$
2	$13 + 9 = 22$
3	13
4	$9 + 5 = 14$
5	5
6	$13 + 13 = 26$
7	$9 + 9 + 9 = 27$
8	$13 + 5 = 18$
9	9

Subtracting 10 from each of these, and taking the largest, we get the largest number that can't be bought exactly, which is 21.

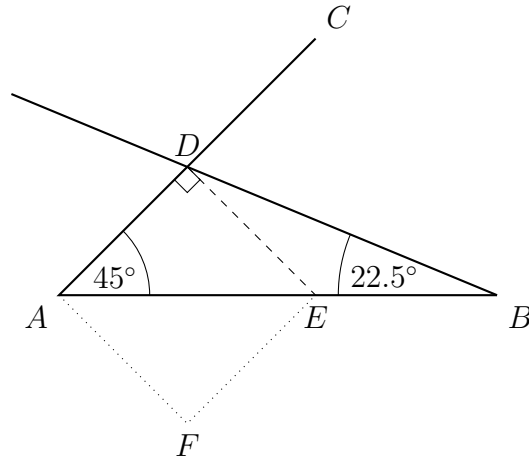
This is known as the Frobenius coin problem (well, I guess here it is the Frobenius donut problem). What we've found is that $\text{Frobenius}(5, 9, 13) = 21$. The problem can be rephrased with any number of donut packet sizes, and as it turns out, if they are in arithmetic progression, as they here, there's an exact formula

$$\text{Frobenius}(a, a + d, \dots, a + nd) = \left(\left\lfloor \frac{a - 2}{n} + 1 \right\rfloor \right) a + (d - 1)(a - 1) - 1,$$

where $\lfloor x \rfloor$ is the largest integer smaller than x . Thus

$$\text{Frobenius}(5, 9, 13) = \left(\left\lfloor \frac{5-2}{2} + 1 \right\rfloor \right) \times 5 + (4-1)(5-1) - 1 = 2 \times 5 + 3 \times 4 - 1 = 21.$$

4. Construct another line, AC at 45° to AB (by bisecting a perpendicular to AB at A). Construct a third line at 22.5° to AB at (B) (by bisecting a perpendicular twice). Produce this line until it intersects with AC at D . Then AD is the length of the side of the square.



Proof:

Drop a perpendicular from D to E on AB (shown with the dashed line in the diagram). Then $\angle AED = 45^\circ$, and by the exterior angle theorem, we can see that $\angle EDB = 22.5^\circ$. Thus $\triangle BDE$ is isosceles, as it has two equal angles, and so $DE = EB$. Thus $AB = AD + DE$, and since $\triangle ADE$ is a right isosceles triangle, it forms one half of a square with AD the diagonal.

To complete construction of the square, we drop perpendiculars to AC at A and ED at E . The point of intersection of these two perpendiculars gives us the location of the fourth vertex of the square, F .

5. We need to count how many subsets T have $x_{min} = n$ for each integer $1 \leq n \leq 10$. For instance, $x_{min} = 10$ for only one subset, $\{10, 11, 12, 13, 14, 15, 16, 17\}$. To create an 8-element subset with $x_{min} = n$, imagine writing the elements of the subset in order. Clearly the first number is n , but we can choose the remaining 7 elements from $(17-n)$ options. Thus the number of subsets with $x_{min} = n$ is given by ${}^{17-n}C_7$.

So the arithmetic mean of the numbers selected is

$$\begin{aligned} \bar{x} &= \frac{1}{24310} (1 \times {}^{16}C_7 + 2 \times {}^{15}C_7 + 3 \times {}^{14}C_7 + \cdots + 10 \times {}^7C_7) \\ &= \frac{1}{24310} (11440 + 12870 + 10296 + 6864 + 3960 + 1980 + 840 + 288 + 72 + 10) \\ &= \frac{48620}{24310} = 2 \end{aligned}$$

Senior Questions

1. (a) Differentiating to find the stationary points we have

$$f'(x) = 3ax^2 + 2bx + c$$

And solving for $3ax^2 + 2bx + c = 0$,

$$\begin{aligned}x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &= \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\end{aligned}$$

- (b) If $b^2 - 3ac < 0$, then the cubic has no stationary points. If $b = 0$, the cubic will have one stationary point if $c = 0$, or two stationary points if a and c have opposite signs.
- (c) Firstly, let's find the coordinates of the point of inflexion.

$$f''(x) = 6ax + 2b$$

So if $f''(x) = 0$, then $x = -\frac{b}{2a}$, which we can see is the average of the x coordinates of the two stationary points.

This occurs because a cubic has rotational symmetry about the point of inflexion. If we re-write the equation of the cubic in terms of the variables $X = x + \frac{b}{2a}$ and $Y = \frac{bc}{3a} - \frac{3b^3}{27a^2}$ (that is, if we translate the coordinate system so that the point of inflexion is at the origin), then the new equation becomes $Y = \alpha X^3 + \beta X$, which we can easily see is an odd function.

2. We use the binomial theorem to expand x^3 . Then

$$\begin{aligned}x^3 &= (\sqrt[3]{10} + \sqrt[3]{6})^3 \\ &= 10 + 3(\sqrt[3]{10})^2\sqrt[3]{6} + 3\sqrt[3]{10}(\sqrt[3]{6})^2 + 6 \\ &= 16 + 3(\sqrt[3]{10} + \sqrt[3]{6})(\sqrt[3]{10})(\sqrt[3]{6}) \\ &= 16 + 3x\sqrt[3]{60} \\ \therefore x^3 - 3x\sqrt[3]{60} &= 16\end{aligned}$$

Now we can consider the polynomial $p(x) = x^3 - 3x\sqrt[3]{60} - 16$. Then x is a zero of p , and we must show that there are no zeros of p that are larger than 4. To do this, we will show that $p(4) > 0$ and p is monotonically increasing for $x \geq 4$. Firstly,

$$p(4) = 64 - 12\sqrt[3]{60} - 16 = 48 - 12\sqrt[3]{60}$$

Since $3^3 < 60 < 4^3$, $\sqrt[3]{60} < 4$ and so $p(4) > 0$. Furthermore, $p'(x) = 3x^2 - 3\sqrt[3]{60} = 3(x^2 - \sqrt[3]{60})$, and if $x > 4$, then $p'(x) > 0$. Thus $p(x)$ has no roots larger than 4. Hence $x < 4$.