MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 14, September 3, 2018

1. Let the middle number be n (usually I find that doing this leads to some simplification in the subsequent algebra). Then the three consecutive cubes are $n-1$, n and $n+1$, so the sum is

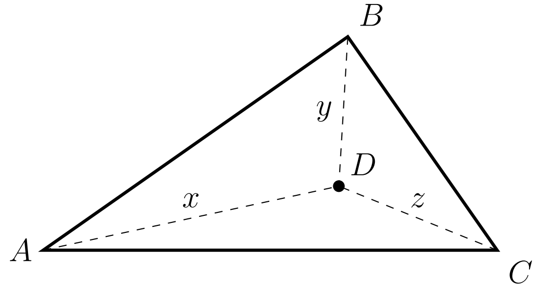
$$\begin{aligned}(n-1)^3+n^3+(n+1)^2 &= (n^3-3n^2+3n-1)+n^3+(n^3+3n^2+3n+1)\\ &= 3n^3+6n\\ &= 3n(n^2+2)\end{aligned}$$

Clearly this sum has a factor of 3, irrespective of the value of n, so we only need to check if $n(n^2+2)$ is a multiple of 6. So drawing up the following table in mod 6, we have

n	n^2+2	$n(n^2+2)$
0	2	0
1	3	3
2	0	0
3	5	3
4	0	0
5	3	3

If $n \equiv 0,2,4 \mod 6$, then the sum is divisible by 18. This means that the first number in the sum must be an odd number.

2. Suppose that the distances from D to the respective vertices are x, y and z, as shown in the diagram.



By the triangle inequality,

$$x + y > AB,$$
$$y + z > BC,$$
$$x + z > AC.$$

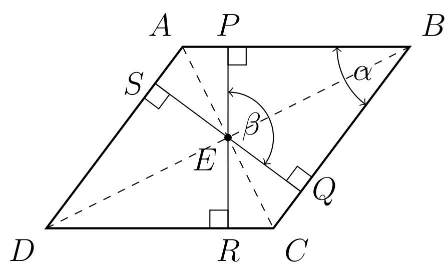
Adding these three inequalities together, we have

$$2x + 2y + 2z > AB + BC + AC,$$

so

$$x + y + z > \frac{1}{2}(AB + BC + AC).$$

3.



Now $\angle EPB = \angle EQB = 90°$, and since the diagonals of a rhombus bisect the angles, $\angle PBE = \angle QBE$. Furthermore, EB is common, so $\triangle EPB \equiv \triangle EQB$ by AAS. Thus $EP = EQ$, as they are corresponding sides in congruent triangles. In a similar fashion, we can show that EP, EQ, ER and ES are all equal.

Now consider the quadrilateral $PBQE$. Let $\angle PBQ = \alpha$ and $\angle PEQ = \beta$. Since $\angle EPB = \angle EQB = 90°$, $\angle PBQ$ and $\angle PEQ$ are supplementary: that is $\alpha + \beta = 180°$. By a similar argument, we can show that $\angle SAP$ and $\angle SEP$ are supplementary. As AD is parallel to BC, $\angle SAP$ and $\angle PBQ$ are co-interior angles. Thus $\angle SAP = \beta$, and hence $\angle SEP = \alpha$, which implies that$\angle SEQ = \alpha + \beta = 180°$, and so SQ is a straight line. In a similar fashion, we can show that PR is a straight line. Thus the quadrilateral $PQRS$ has diagonals SQ and PR that are equal and bisect each other, and hence is a rectangle.

4. First, let's note that to end up heads up, the coin would have to be flipped an odd number of times. Take the nth coin in the line, how many times does it get flipped? Every coin gets flipped on the first pass, but only every second on the second pass—that is, only those coins whose position is divisible by 2. Similarly, on the 3rd pass only those coins sitting on multiples of 3 get flipped. So the nth coin will get flipped on the mth pass if m is a factor of n.

   If n is written in terms of its prime factorisation, $n = p_1^{d_1} p_2^{d_2} \dots p_k^{d_k}$, then any factor must be able to be written as $n = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$ where $0 \leq c_i \leq d_i$. Thus the number of factors of n is the product of the number of possibilities for choosing each c_i. There are $d_i + 1$ choices for c_i $(0, 1, \ldots, d_i)$ so in total n has $(d_1+1)(d_2+1)(d_3+1)\ldots(d_k+1)$ factors. (You may recall the 'tau' function, $\tau(n)$, from a problem sheet last year.) For n to have an odd number of factors then every d_i must be even, implying that n is a perfect square. So the coins that end up heads up are those that are positioned at a perfect square, i.e. $1, 4, 9, 16, \ldots$.

The square root of 1000 is $31.62277\ldots$, so 31^2 is the largest perfect square less than 1000. Thus only 31 coins out of the 1000 end up heads up.

5. Note that $792 = \text{lcm}(88, 99)$. Then

$$\begin{aligned}\left((88!)^{1/88}\right)^{729} &= (88!)^9\\ \left((99!)^{1/99}\right)^{729} &= (99!)^8\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{(99!)^8}{(88!)^9} &= \left(\frac{99!}{88!}\right)^8 \frac{1}{88!}\\ &= \frac{(99 \times 98 \times \ldots \times 89)^8}{88 \times 87 \times \ldots \times 2 \times 1}.\end{aligned}$$

If you consider this fraction, we can see that there are 88 numbers in both numerator and denominator. However, every number in the numerator is larger than every number in the denominator. So this fraction is greater than one. Hence

$$(99!)^8 > (88!)^9.$$

Taking the 792nd root of both sides (which is OK because both numbers are positive) we have

$$(99!)^{1/99} > (88!)^{1/88},$$

or

$$\sqrt[99]{99!} > \sqrt[88]{88!}.$$

Senior Questions

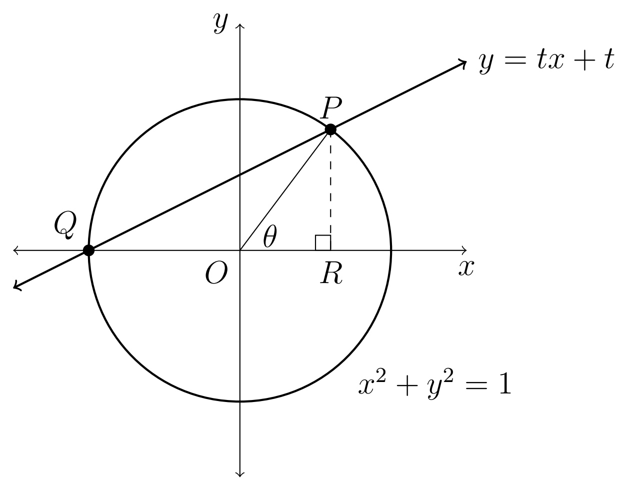
1. (a) Substituting $y = tx + t$ into $x^2 + y^2 = 1$, we have

$$\begin{aligned}x^2 + (tx+t)^2 &= 1\\ (1+t^2)x^2 + 2t^2x + t^2 - 1 &= 0\\ \therefore x &= \frac{-2t^2 \pm \sqrt{4t^4 - 4(1+t^2)(t^2-1)}}{2(1+t^2)}\\ &= \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^4-1)}}{2(1+t^2)}\\ &= \frac{-t^2 \pm \sqrt{1}}{1+t^2}\\ &= -1,\ \frac{1-t^2}{1+t^2}\end{aligned}$$

Clearly, $x = -1$ corresponds to the point Q. Substituting $x = \frac{1-t^2}{1+t^2}$ into $y = tx+t$, we obtain $y = \frac{2t}{1+t^2}$. Thus the coordinates of P are $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

(b) Typically, we parametrise the unit circle in terms of θ, where θ is the angle between the ray OP and the positive x axis. Then we have $x = \cos\theta$ and $y = \sin\theta$. Here, we have used an alternative parametrisation using the point of intersection of the line $y = tx + t$ and the unit circle (at least for the part of the unit circle lying in the first quadrant).

As you can see in the diagram below, $\triangle OPQ$ is isosceles, and thus $\angle OPQ = \angle OQP$. Furthermore, by the exterior angle theorem, $\angle OPQ + \angle OQP = \theta$. Hence $\angle OQP = \frac{\theta}{2}$. But $\angle OQP$ is also the angle of incidence of the line $y = tx+t$, which has gradient t. Hence $t = \tan\frac{\theta}{2}$.



(c) If we drop a perpendicular from P to the x axis at R, then $\triangle OPR$ is a right angle triangle with sides in the ratio

$$\frac{1-t^2}{1+t^2} : \frac{2t}{1+t^2} : 1.$$

Let $t \in (0,1)$ be a rational number. That is, $t = \frac{p}{q}$, where p and q are positive integers (in lowest terms) and $p < q$. Then the sides of $\triangle OPR$, expressed in terms of p and q, are

$$\frac{1-(p/q)^2}{1+(p/q)^2} : \frac{2(p/q)}{1+(p/q)^2} : 1.$$

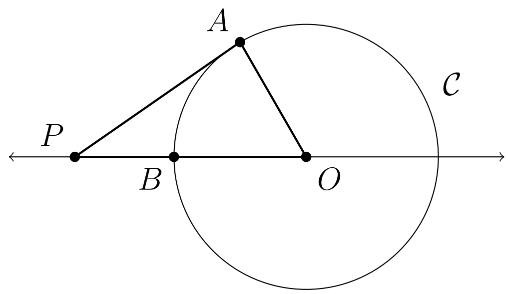
Since we want the triangle to have integer sides, we multiply this ratio by $q^2\left(1+\frac{p^2}{q^2}\right)$ to clear out the denominators of the two fractions. We then obtain the ratio

$$q^2 - p^2 : 2pq : p^2 + q^2,$$

which gives us a Pythagorean triple. Each rational value of $t \in (0,1)$ corresponds to a different Pythagorean triple, and since there are an infinite number of rational numbers in the interval $(0,1)$, there are an infinite number of right-angled triangles with integer sides.

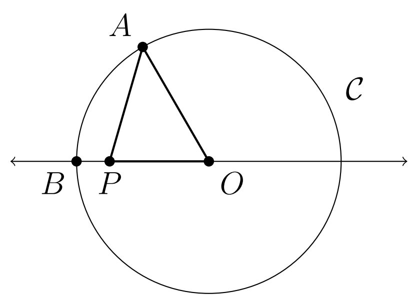
2. (a) Let $\mathcal{C}$ be a circle centred at O. Let P be a point lying in the plane, and let A be an arbitrary point on $\mathcal{C}$. There are two possibilities to consider: (i) P lies outside the circle or (ii) P lies inside the circle.

Firstly, suppose that P lies outside the circle. Let B be the point of intersection between $\mathcal{C}$ and the line through OP that lies in-between O and P, as shown in the diagram.



By the triangle inequality, $AP + OA \geq OP = PB + OB$. But $OA = OB$, as they are both radii of the circle $\mathcal{C}$. Thus $AP \geq PB$, and since A is an arbitrary point on $\mathcal{C}$, this means that B is the point closest to P.

Now suppose that P lies inside the circle. Let B be the point of intersection between $\mathcal{C}$ and the line through OP such that P lies between B and O, as shown in the diagram.



We need to show that $AP \geq BP$. By the triangle inequality, $OP + AP \geq OA$. But $OA = OB = OP + BP$. Thus $OP + AP \geq OP + BP$, which implies that $AP \geq BP$, as required.

(b) We will use the fact that, for any $z \in \mathbb{C}$, $|z|^2 = z\overline{z}$. Thus

$$\begin{aligned}|z - w|^2 &= (z - w)(\overline{z} - \overline{w}) \\ &= z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} \\ &= |z|^2 - z\overline{w} - w\overline{z} + |w|^2 \\ &= r^2 + 1 - (z\overline{w} + w\overline{z})\end{aligned}$$

Now in polar form,

$$\begin{aligned}
z\overline{w} + w\overline{z} &= r(\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi) + (\cos\phi + i\sin\phi)r(\cos\theta - i\sin\theta)] \\
&= r[(\cos\theta\cos\phi - i\sin\phi\cos\theta + i\sin\theta\cos\phi + \sin\theta\sin\phi) \\
&\qquad + (\cos\theta\cos\phi - i\sin\theta\cos\phi + i\sin\phi\cos\theta + \sin\theta\sin\phi)] \\
&= 2r(\cos\theta\cos\phi + \sin\theta\sin\phi) \\
&= 2r\cos(\theta - \phi).
\end{aligned}$$

So

$$|z-w|^2 = r^2 + 1 - 2r\cos(\theta - \phi).$$

Thus $|z-w|^2$ is minimized when $\cos(\theta-\phi) = 1$. That is, when $\phi = \theta$.