MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 16, September 17, 2018

1. It requires two people to shake hands. According to the guests' claims, we see that there have been exactly $5 \times 11 = 55$ instances of people taking part in one half of a handshake. As this is not an even number, it cannot be twice the total number of handshakes. Thus someone is lying.

2. In the $3 \times 3 \times 3$ cube a single die could be located at a vertex, an edge, or in the centre of a face. A vertex die contributes the numbers on three of its faces to the total; an edge die contributes two; and a central die contributes only one. There are eight vertex dice; twelve edge dice and 6 central dice. Thus the smallest sum is $8 \times (1 + 2 + 3) + 12 \times (1 + 2) + 6 \times 1 = 90$.

3. Let the number we are seeking be x. We will calculate the digits of x by working from the leftmost digit to the right. If we fix the first digit, there are $9! = 362\,880$ ways to arrange the remaining 9. So there are 362 880 numbers in the list starting with '0', then another 362 880 starting with '1' and so on. Now $\lceil 999\,999/362\,880 \rceil = 3$, so

$$2 \times 9! < 1\,000\,000 < 3 \times 9!.$$

Thus the first digit of x is the third digit in the list $0, 1, \ldots, 9$, which is 2. So x starts with a 2.

If the first two digits are fixed, there are $8! = 40\,320$ ways to arrange the remaining digits, and we find that

$$2 \times 9! + 6 \times 8! < 1\,000\,000 < 2 \times 9! + 7 \times 8!$$

Now 2 has already been used for the first digit, so the second digit is the 7th number remaining from $0, 1, 3, \ldots, 9$, which is 7.

For the third digit, we find that

$$2 \times 9! + 6 \times 8! + 6 \times 7! < 1\,000\,000 < 2 \times 9! + 6 \times 8! + 7 \times 7!,$$

and so the third digit is 8. Continuing in this fashion, we eventually find that x is 2 783 915 460.

4. This is basically a proof by exhaustion of cases. A two-digit narcissistic number with digits ab must satisfy

$$a^2 + b^2 = 10a + b,$$

or

$$b^2 - b + (a^2 - 10a) = 0.$$

We can consider this as a quadratic in b, with discriminant

$$\Delta = 1 - 4(a^2 - 10a) = 101 - 4(a-5)^2.$$

If a is an integer between 1 and 9, we obtain the following values for Δ:

a	Δ
1	37
2	65
3	85
4	97
5	101
6	97
7	85
8	65
9	37

As none of these values is a perfect square, b is an irrational number in all cases. So there are no 2-digit narcissistic numbers.

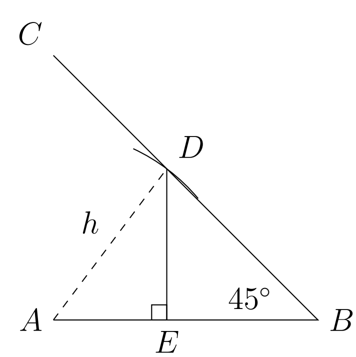
5. (a) Suppose that we are given the length of the hypotenuse h and the sum of the two short sides, s.

(i) Construct a line AB equal to s.

(ii) Construct a ray, BC, at an angle of 45° to AB at B.

(iii) Using the compasses, draw an arc with radius h centered at A. Let D be the point where this arc intersects BC. (NB: two possible positions for D.)

(iv) Drop a perpendicular from D to AB. Let the foot of this perpendiular be E. Then $\triangle ADE$ is the desired triangle.



<u>Proof:</u> Clearly $\triangle ADE$ is a right-angled triangle with hypotenuse h. Furthermore, $\triangle BED$ is an isosceles right-triangle, and hence $DE = EB$. Thus $AE + DE = AB = s$, as required.

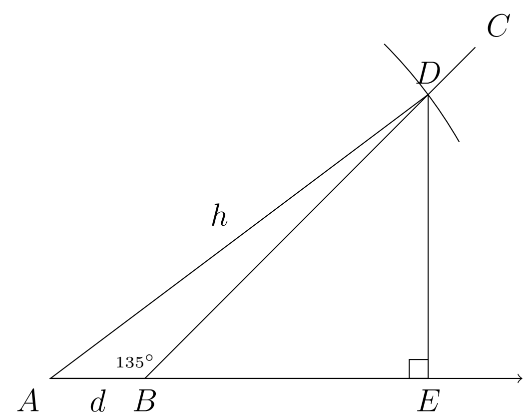
(b) Suppose that we are given the length of the hypotenuse h and the difference of the two short sides, d.

(i) Construct a line segment AB with length d.

(ii) Construct a ray, BC, at an angle of $135°$ to AB at B.

(iii) Using the compasses, find a point D on BC that is a distance of h from A.

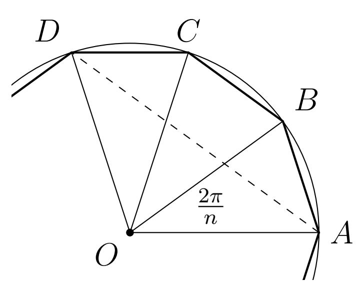
(iv) Extend AB and drop a perpendicular from D which meets AB at E. Then $\triangle ADE$ is the desired triangle.



Proof: Clearly $\triangle ADE$ is a right-angled triangle with hypotenuse h. Since $\angle ABD = 135°$, $\angle DBE = 45°$. Thus $\triangle BED$ is a right isosceles triangle and $BE = ED$. Hence AB is the difference between DE and AE, as required.

Senior Questions

1. Drawing the radii from the centre of the circle to each of the vertices of the regular n-gon makes wedges each with an angle of $\frac{2\pi}{n}$ at the centre.



It can be shown that the side length of the n-gon is $2r\sin\left(\frac{\pi}{n}\right)$, where r is the radius of the circle. The side AD makes up a triangle across three of these wedges, so its length is $2r\sin\left(\frac{3\pi}{n}\right)$. Since AD is the side length plus the radius,

$$2r\sin\left(\frac{\pi}{n}\right) + r = 2r\sin\left(\frac{3\pi}{n}\right).$$

We can cancel the common factor of r and expand the right hand side using the trigonometric identity

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

We then obtain a cubic polynomial in $\sin\left(\frac{\pi}{n}\right)$. Letting $x = \sin\left(\frac{\pi}{n}\right)$, the cubic can be written as

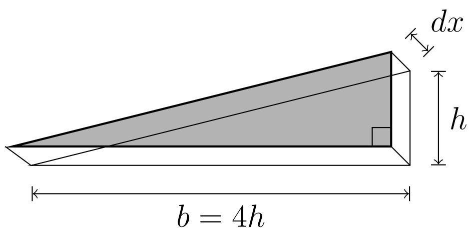
$$8x^3 - 4x + 1 = 0.$$

Using the remainder theorem, we can easily confirm that $x = \frac{1}{2}$ is a solution to this polynomial. Then using polynomial long division, we can show that

$$8x^3 - 4x + 1 = \left(x - \frac{1}{2}\right)(8x^2 + 4x - 2).$$

Solving the quadratic with the formula, we obtain the three solutions $x = \frac{1}{2}$, and $x = \frac{-1 \pm \sqrt{5}}{4}$. From this, we can use the calculator to check that the values of n that work are $n = 6$ and $n = 10$.

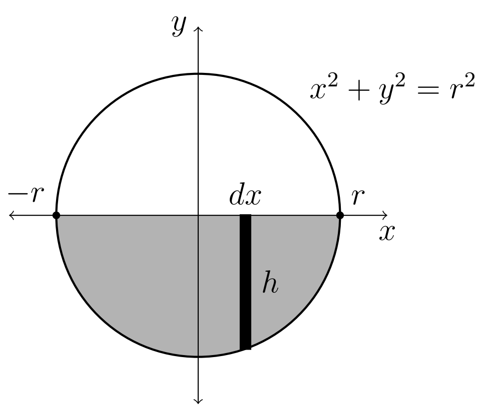
2. Let V be the volume of water in the glass. We can calculate V using integration by slices. If we take slices parallel to the axis of symmetry of the cylinder, we get a series of similar triangular prisms, where the base is 4 times the height of the triangle.



If we let the depth of the prism be dx then the volume of the prism (dV) is given by

$$\begin{aligned} dV &= \frac{1}{2} bh\, dx \\ &= 2h^2\, dx \end{aligned}$$

If we look at the base of the cylinder, we can see that h will vary with x according to the equation $h = \sqrt{r^2 - x^2}$.



Consequently,

$$\begin{aligned} V &= \int_{-r}^{r} 2h^2\, dx \\ &= \int_{-r}^{r} 2(r^2 - x^2)\, dx \\ &= 4 \int_{0}^{r} r^2 - x^2\, dx \quad \text{(by symmetry)} \\ &= 4 \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= \frac{8r^3}{3} \end{aligned}$$