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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 4, June 4, 2018

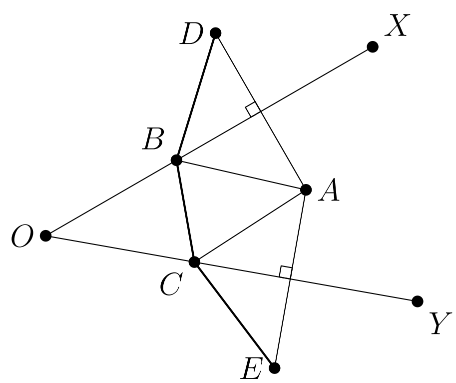
1. Since x is an integer, x^2 is the product of even powers of 2 and 3, and hence y^3 is also a product of even powers of 2 and 3. Then y^3 can be 1, 2^6, 2^{12}, 3^6, 3^{12}, $2^6 \cdot 3^6$, $2^6 \cdot 3^{12}$, $2^{12} \cdot 3^6$ or $2^{12} \cdot 3^{12}$. For each of these y values, there is one value of x. Hence there are nine solutions altogether.

2. Write $\frac{11}{42}$ as a simple continued fraction. That is,

$$\begin{aligned}\frac{11}{42} = \frac{1}{\frac{42}{11}} &= \frac{1}{3+\frac{9}{11}} \\ &= \frac{1}{3+\frac{1}{\frac{11}{9}}} = \frac{1}{3+\frac{1}{1+\frac{2}{9}}} \\ &= \frac{1}{3+\frac{1}{1+\frac{1}{\frac{9}{2}}}} = \frac{1}{3+\frac{1}{1+\frac{1}{4+\frac{1}{2}}}}\end{aligned}$$

Then $a + b + c + d = 3 + 1 + 4 + 2 = 10$.

3. We use the method of reflection.



Let A be a point lying inside the angle XOY and let B and C be points on OX and OY as shown in the digram. Let D and E be the reflection of the point A in the lines OX and OY, respectively. Then $\triangle ABD$ and $\triangle ACE$ are both isosceles with $AC = CE$ and $AB = BD$. Thus the path from D to E via B and C is equal in length to the perimeter of $\triangle ABC$. Hence this length is minimised when $DBCE$ is a straight line.

4. The sum of the digits $1, 2, 3, \ldots, 9$ is 45 $[(1+9)+(2+8)+\ldots+5]$. Also recalling that if we have a sum like $\sum_{k=0}^{n}(a+k) = a(n+1) + \sum_{k=0}^{n} k$, then the required sum is

$$\begin{aligned}
\sum_{a=0}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9}\sum_{d=0}^{9}(a+b+c+d) &= \sum_{a=0}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9}\left(10(a+b+c)\sum_{d=0}^{9} d\right) \\
&= \sum_{a=0}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9}(45+10a+10b+10c) \\
&= \sum_{a=0}^{9}\sum_{b=0}^{9}\left(10(45+10a+10b)+10\sum_{c=0}^{9} c\right) \\
&= \sum_{a=0}^{9}\sum_{b=0}^{9}(450+100a+100b+450) \\
&= \sum_{a=0}^{9}\left(10(900+100a)+100\sum_{b=0}^{9} b\right) \\
&= \sum_{a=0}^{9}(9000+1000a+4500) \\
&= 10 \times 13\,500 + 1000 \times 45 \\
&= 180\,000
\end{aligned}$$

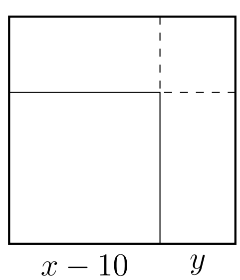
Senior Questions

1. Firstly, we complete the square in a slightly unusual way.

$$\begin{aligned}
x^2 - 19x + 94 &= x^2 - 20x + 100 + x - 6 \\
&= (x-10)^2 + x - 6
\end{aligned}$$

Then $(x-10)^2$ is a perfect square whenever x is an integer.

Consider the following diagram



We want to make

$$(x - 10 + y)^2 = x^2 - 20x + 100 + x - 6,$$

where x and y are integers. Thus

$$\begin{aligned} y^2 + 2(x-10)y &= x - 6 \\ y^2 + 2xy - 20y &= x - 6 \\ y^2 - 20y + 6 &= x(1-2y) \end{aligned}$$

So

$$x = \frac{y^2 - 20y + 6}{1 - 2y}.$$

Using polynomial long division, we find that

$$x = -\frac{y}{2} + \frac{39}{4} - \frac{15}{4}\left(\frac{1}{1-2y}\right).$$

We multiply this by 4 to obtain

$$4x = -2y + 39 - \frac{15}{1-2y}.$$

This can be made simpler if we re-write it as

$$4x = 1 - 2y + 38 - \frac{15}{1-2y},$$

and then make the substitution $w = 1 - 2y$, so then

$$4x = w - \frac{15}{w} + 38.$$

If we want to have x an integer, then w must be a factor of 15. Since there are a finite number of integer solutions for w $(\pm 1, \pm 3, \pm 5, \pm 15)$, we simply need to find the one that gives the largest value of x. If we do this, we find that $x = 13$.

2. We use the method of reflection again. Let B' be the reflection of the point B in the river. Then the length L is equal to the path from A to B' via E, which is minimized when AEB' is a straight line. In this case, the distance is 15 km (a nice 3-4-5 right triangle).