MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 6, June 18, 2018

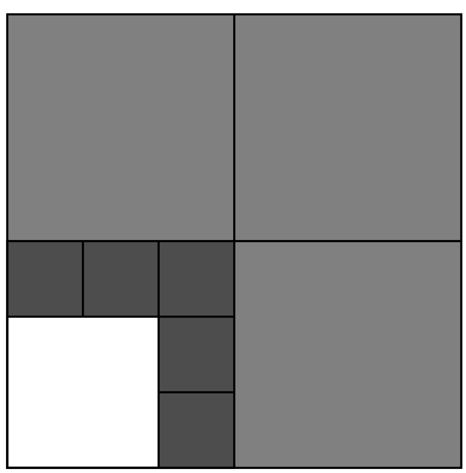
1. Firstly, we note that $2x + 5y \neq 0$. Then

$$\begin{aligned} \frac{x+3y}{2x+5y} &= \frac{4}{7} \\ 7(x+3y) &= 4(2x+5y) \\ 7x + 21y &= 8x + 20y \\ x &= y \end{aligned}$$

The solution is then $x = y \neq 0$.

2. The four digit numbers that satisfy the first equation are 1064, 1164, 1264 and so on. Of these, only 1764, 3364, 8464 are square. These satisfy the first equation for $n = 17$, 33 and 84 respectively. Then $201 \times 84 + 64 = 16\,948$ which is larger than 4 digits, and $201 \times 33 + 64 = 6697$ which isn't square. So $n = 17$.

3. The solution looks like this:



4. (a) Consider the integers in mod 4. Then it is clear that the only squares in mod 4 are 0 and 1.

   (b) Let the three integers be x, y and z and consider them in mod 4. An odd integer is congruent to either 1 or 3 mod 4. So if all three integers are odd, we can draw up the following table where all calculations have been carried out in mod 4:

x	y	z	$x+y$	$x+z$	$y+z$
1	1	1	2	2	2
1	1	3	2	0	0
1	3	3	0	0	2
3	3	3	2	2	2

We know from part (a) that squares are either 0 or 1 mod 4, so none of these combinations works.

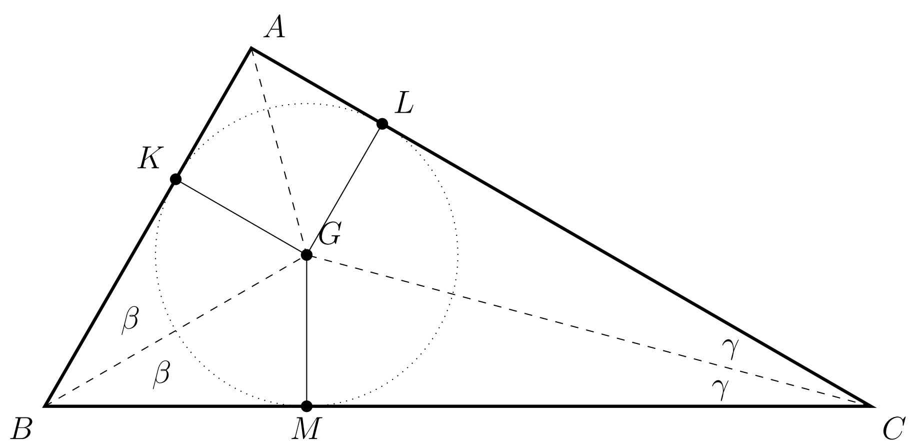
Without loss of generality, suppose that x is even but both y and z are odd. Then the table is as follows:

x	y	z	$x+y$	$x+z$	$y+z$
0	1	1	1	1	2
0	1	3	1	3	0
0	3	3	3	3	2
2	1	1	3	3	2
2	1	3	3	1	0
2	3	3	1	1	2

Once again, we see that this does not work if two of the integers are odd. Consequently, at most one of the integers is odd. (It can be seen that $x, y \equiv 0$ and $z \equiv 1$ or $x, y \equiv 2$ and $z \equiv 3$ mod 4, for instance, might work.)

(c) Try 19, 30 and 6.

5. Let KG, LG and GM be perpendiculars from G to AB, AC and BD, respectively. Then $\triangle BKG$ and $\triangle BGM$ are two right triangles with a smaller angle and a hypotenuse in common, so $\triangle BKG \equiv \triangle BGM$. Thus $GK = GM$. By a similar argument, it can be shown that $GM = GL$. Consequently, $\triangle GAK \equiv \triangle GAL$, and so GA bisects $\angle BAC$, as required.



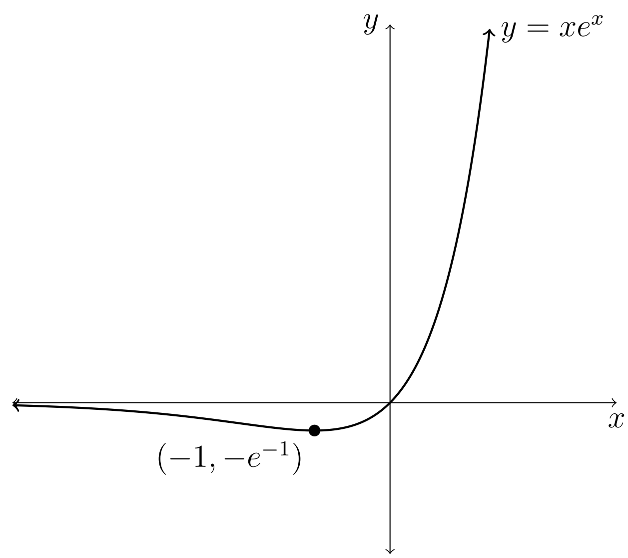
As you can see from the proof, the points K, L, and M are equidistant from G. Thus G is the centre of a circle that can be drawn inside the triangle and tangent to each side, which is called the inscribed circle.

Senior Questions

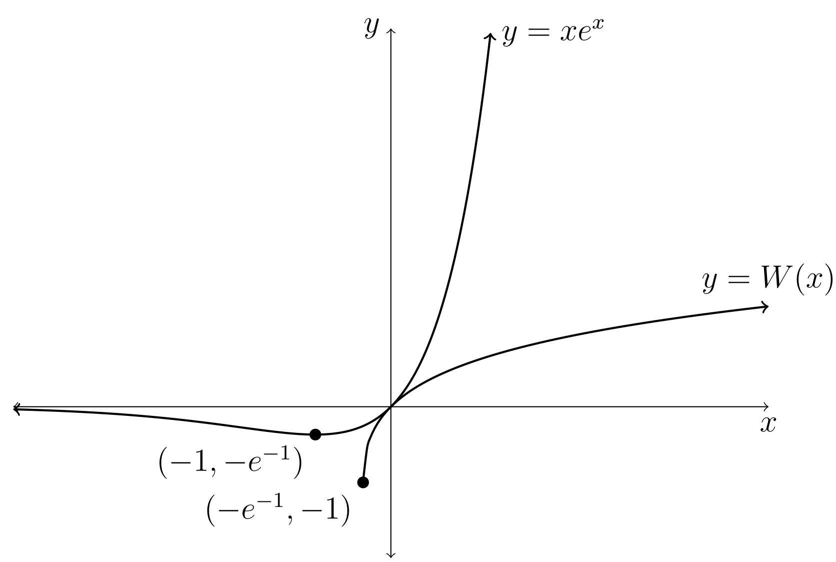
1. Use mathematical induction.

2. Recall that $1^2+2^2+3^2+\ldots+n^2 = \frac{n(2n+1)(n+1)}{6}$. Then $\displaystyle\lim_{n\to\infty} \frac{1^2+2^2+3^2+\ldots+n^2}{n^3} = \frac{1}{3}$.

3. (a) The graph should be as follows:



(b) And the graph with the inverse function shown is as follows:



(c) Let $y = W(x)$. Then by definition, $x = ye^y$. Differentiating implicitly with respect to x, we have

$$1 = \frac{dy}{dx}e^y + ye^y\frac{dy}{dx}$$
$$1 = e^y(1+y)\frac{dy}{dx}$$

But $e^y = \frac{x}{y}$, and so

$$\begin{aligned} 1 &= \frac{x(1+y)}{y} \cdot \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{y}{x(1+y)} \end{aligned}$$

And since $y = W(x)$, the result follows.