MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 9, July 30, 2018

1. The angles in the triangle are, in ascending order 2α, 3α, 4α for some value of α. By the angle sum of the triangle,

$$\begin{aligned} 2\alpha + 3\alpha + 4\alpha &= 180^circ \\ 9\alpha &= 180^\circ \\ \therefore \alpha &= 20^\circ \end{aligned}$$

   Thus the largest angle is 80°.

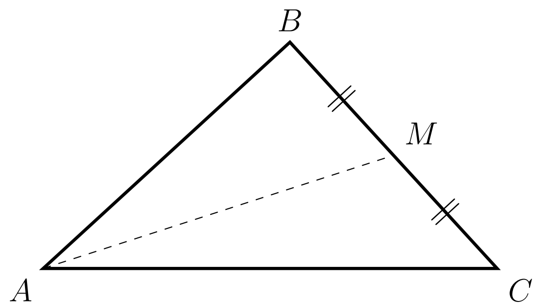
2. You can work this out on your calculator using the $\log_{10}$ button.

$$\begin{aligned} \log_{10}(125)^{100} &= 100\log_{10}(125) \\ &= 209.69\ldots \end{aligned}$$

   Now we can tell the number of digits of a number n by considering the integer part of $\log_{10}(n)$. If $\lfloor \log_{10}(n) \rfloor = k$, then n has $k+1$ digits, so we can see that 100^{125} has 210 digits.

3. Applying the triangle inequality to $\triangle AMB$, we have

$$\begin{aligned} AM &< AB + BM \\ \therefore AM &< AB + \frac{1}{2}BC. \end{aligned}$$



Similarly, applying the triangle inequality to $\triangle AMC$, we have

$$AM < AC + \frac{1}{2}BC.$$

If we add these two inequalities, we have

$$2AM < AB + BC + AC.$$

Thus

$$AM < \frac{1}{2}(AB + BC + AC).$$

4. We can write α as

$$\begin{aligned} \alpha &= \frac{1}{1+\alpha} \\ \alpha(1+\alpha) &= 1 \\ \alpha^2 + \alpha - 1 &= 0 \end{aligned}$$

This is just a quadratic in α, so

$$\begin{aligned} \alpha &= \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} \\ &= \frac{-1 \pm \sqrt{5}}{2}. \end{aligned}$$

Clearly, $\alpha > 0$, so we take the positive square root, and thus $\alpha = \frac{-1+\sqrt{5}}{2}$.

5. (a) Recall that $\gcd(a + mb, b) = \gcd(a, b)$. So if we have $\gcd(m, n)$ with $m > n$ and we divide m by n to get a remainder r, then $\gcd(m, n) = \gcd(n, r)$. (This idea is the basis of the Euclidean algorithm.) Thus

$$\begin{aligned} 2^{50} + 1 &= (2^{20} + 1)(2^{30} - 2^{10}) + \underline{2^{10} + 1} \\ 2^{20} + 1 &= (2^{10} + 1)(2^{10} - 1) + \underline{2} \\ 2^{10} + 1 &= (2)(2^9) + \underline{1} \\ 2 &= 2 \times 1 + \underline{0} \end{aligned}$$

Working backwards, we can see that $\gcd(2^{50} + 1, 2^{20} + 1) = 1$.

(b) I think the simplest way to do this is to consider the sum of two nth powers. If n is an odd number,

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + y^{n-1})$$

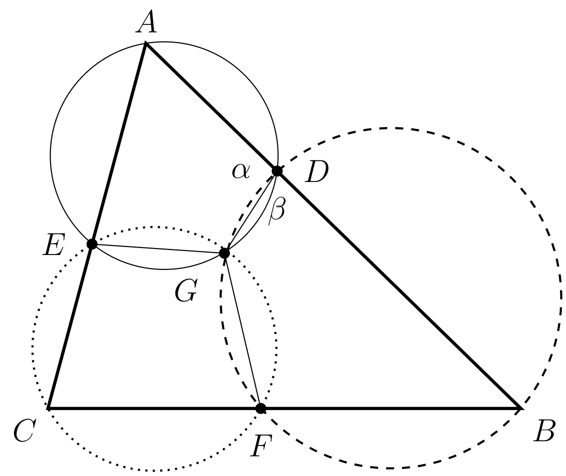
So if m and n are both odd, then

$$\begin{aligned} 2^m + 1 &= (2 + 1)(2^{m-1} - 2^{m-2} + 2^{m-3} - \ldots + 1) \\ 2^n + 1 &= (2 + 1)(2^{n-1} - 2^{n-2} + 2^{n-3} - \ldots + 1) \end{aligned}$$

We can see clearly that these numbers have a common factor of three. Thus the common divisor must be a multiple of three.

Senior Questions

1. We do this by letting the circles ADE and BDF intersect at a point G. We will then prove that $ECFG$ is a cyclic quadrilateral.



Join the lines DG, GE and GF. Let $\angle ADG = \alpha$ and $\angle BDG = \beta$. Then α and β are complementary angles.

Since $BDGF$ is a cyclic quadrilateral, $\angle BFG = \alpha$ and so $\angle GFC = \beta$. Similarly, $\angle AEG = \beta$ and thus $\angle GEC = \alpha$. Thus $\angle GEF + \angle GFC = 180°$, which means that $ECFG$ is a cyclic quadrilateral. Consequently, the points E, C, F and G are concyclic (that is, they all lie on the same circle).

2. If $\cos(A + B) + \sin(A - B) = 0$, then

$$\begin{aligned}\cos A \cos B - \sin A \sin B + \sin A \cos B - \sin B \cos A &= 0\\ \cos A(\cos B - \sin B) + \sin A(\cos B - \sin B) &= 0\\ (\cos A + \sin A)(\cos B - \sin B) &= 0\end{aligned}$$

So either $\cos A + \sin A = 0$ or $\cos B - \sin B = 0$.
In the first case, $\tan A = -1$, so

$$A = -\frac{\pi}{4} + k\pi = \frac{(4k-1)\pi}{4}.$$

In the second case, $\tan B = 1$, hence

$$B = \frac{\pi}{4} + k\pi = \frac{(4k+1)\pi}{4}.$$

To solve $\cos(n\theta) + \sin(m\theta) = 0$, let

$$\begin{aligned}A + B &= n\theta\\ A - B &= m\theta\end{aligned}$$

and solve simultaneously to obtain $A = \dfrac{(n+m)\theta}{2}$ and $B = \dfrac{(n-m)\theta}{2}$. Consequently,

$$\begin{aligned}\frac{(n+m)\theta}{2} &= \frac{(4k-1)\pi}{4}\\ \theta &= \frac{(4k-1)\pi}{2(n+m)}, \quad \text{if } n \neq -m.\end{aligned}$$

Or

$$\begin{aligned}\frac{(n-m)\theta}{2} &= \frac{(4k+1)\pi}{4}\\ \theta &= \frac{(4k+1)\pi}{2(n-m)}, \quad \text{if } n \neq m.\end{aligned}$$