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2011 University of New South Wales School Mathematics Competition [1]

Junior Division – Problems and Solutions

Problem 1

A second-cousin prime n-tuple is defined as a set of n prime numbers $\{p, p+6, \ldots p+6(n-1)\}$ with common difference six. Each number in the set is a prime and consecutive members of the set differ by six. For example 2011 is a member of a second-cousin prime 2-tuple.

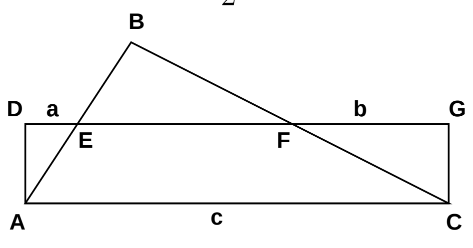
Show that there is one and only one second-cousin prime 5-tuple and there are no second-cousin prime 6-tuples.

Solution 1

Clearly if p is not equal to five and is a member of a second-cousin prime n-tuple then the last digit of p must be one of one, three, seven or nine. Suppose it ends in one, then the next member of the second-cousin prime n-tuple ends with a seven, the next member a three, the next member a nine and then the next number that differs by six ends in a five and is therefore non-prime. Thus there are no second-cousin prime n-tuples with $n > 4$ if the first prime in the set is not equal to five. It remains to consider a second-cousin prime n-tuple starting with $p = 5$. By construction the largest second-cousin prime n-tuple is the second-cousin prime 5-tuple $(5, 11, 17, 23, 29)$ and there are no second-cousin prime 6-tuples.

Problem 2

In the figure below $AC = c$ and the points E and F lie on the line $\overline{DG}$ with $DE = a$ and $FG = b$. Show that if the area of the triangle ABC is equal to the area of the rectangle $ADGC$ then $a + b = \dfrac{c}{2}$.



Solution 2

Consider a line perpendicular to AC that passes through B. Let H denote the point where the line intersects AC and let I denote the point where the line intersects DG.

[1] The problems and solutions were compiled, created, refined with contributions from David Angell, Chris Angstmann, Peter Brown, David Crocker, Bruce Henry (Director), David Hunt and Dmitriy Zanin.

The area of the triangle ABC is given by

$$\text{Area}\triangle ABC = \frac{1}{2}c|BH|,$$

and the area of the rectangle is given by

$$\text{Area}\square ADGC = |IH|c$$

thus the areas are equal if

$$|BH| = 2|IH| \Rightarrow \frac{|BH|}{|IH|} = 2.$$

Note the relation

$$|BH| = |BI| + |IH| \Rightarrow \frac{|BH|}{|IH|} = \frac{|BI|}{|IH|} + 1 \Rightarrow \frac{|BI|}{|IH|} = 1.$$

Note the equal ratios

$$\frac{c-(a+b)}{c} = \frac{|BI|}{|BH|} = \frac{|BI|}{2|IH|} = \frac{1}{2}$$

so that

$$a + b = \frac{c}{2}.$$

Problem 3
Classic unrelated problems with flipping coins.

1. A sequence of zeros and ones is constructed by flipping a coin and assigning zeros to heads and ones to tails. If the coin is unbiased then heads and tails have the same probability of occurrence and the random sequence is said to be uniformly random. If the coin is biased then heads and tails do not have the same probability of occurrence. How can you construct a uniformly random sequence of zeros and ones by flipping a coin with a small but unknown bias?

2. You flip one hundred coins on a table top with your eyes blindfolded and you are reliably informed that thirteen landed heads up and the remainder landed heads down. Supposing you remain blindfolded, how can you sort the coins into two groups so that there are the same number showing heads up in each group? You may turn coins over but you must not discard coins.

Solution 3

1. Flip the coin with the unknown bias to construct a sequence $S_0, S_1, S_2, S_3 \ldots$ where S_j is one of heads or tails. Now construct the sequence of pairs $(S_0, S_1), (S_2, S_3), \ldots$. If (S_k, S_{k+1}) is $(heads, tails)$ then label it a zero and if it is $(tails, heads)$ label it a one. Discard the pairs $(heads, heads)$ and $(tails, tails)$. The resulting sequence of zeros and ones will be uniformly random. This follows since if p is the probability of heads and q is the probability of tails and the coin is biased then $p^2 \neq q^2$ but $pq = qp$.

2. Select thirteen coins at random and move (slide) them into a separate group. There will then be $n \leq 13$ coins heads up and $13 - n$ coins heads down in the separated group and there will be $13 - n$ coins heads up in the remaining group. Now turn all coins over in the separated group to have $13 - n$ coins heads up.

Problem 4

Six hundred and sixty-six students sit for a prestigious mathematics contest. It is known that all of the students who sit the exam attend an all girls school and/or play sport on the weekend, and/or play a musical instrument. One hundred and eleven of the students attend an all girls school and two hundred and twenty-two attend an all boys school. Four hundred and forty-four of the students play musical instruments and five hundred and fifty-five of the students play sport on the weekend. Seventy-seven of the students attend an all girls school and play sport on the weekend. Eighty-eight of the students attend an all girls school and play a musical instrument. Three hundred and thirty-three of the students play a musical instrument and play sport on the weekend. Of the students who attend an all boys school thirty-three of them do not both play sport on the weekend and play a musical instrument. How many of the students attend a co-ed school, play sport on the weekend and play a musical instrument?

Solution 4

Let N denote the total number of students who sat the competition, $N(G)$ the number who attended an all girls school, $N(B)$ the number who attend an all boys school, $N(C)$ the number who attend a co-ed school, $N(M)$ the number who play a musical instrument, and $N(S)$ the number who play sport on the weekends. Let $N(S \cap M)$ denote the number who play sport on the weekends and play a musical instrument etc. The following relations hold

$$\begin{aligned}
N &= N(G) + N(M) + N(S) \\
&\quad -N(G \cap M) - N(G \cap S) - N(M \cap S) \\
&\quad +N(G \cap M \cap S) \\
N(B) &= N(B \cap M \not\cap S) + N(B \cap S \not\cap M) + N(B \cap M \cap S) \\
N(M \cap S) &= N(G \cap M \cap S) + N(B \cap M \cap S) + N(C \cap M \cap S)
\end{aligned}$$

Using the numbers given we have

$$\begin{aligned}
666 &= 111 + 444 + 555 - 88 - 77 - 333 + N(G \cap M \cap S) \\
222 &= 33 + N(B \cap M \cap S) \\
333 &= N(G \cap M \cap S) + N(B \cap M \cap S) + N(C \cap M \cap S)
\end{aligned}$$

It is then a simple matter to solve for the number of students who attend a co-ed school, play sport on the weekend and play a musical instrument, $N(C \cap M \cap S) = 90$.

Problem 5

Show that if a, b, and c are positive integers with $\frac{a}{b} < \sqrt{c}$ and $c > 1$ then

$$\frac{a + bc}{a + b} > \sqrt{c}.$$

Solution 5

$$
\begin{aligned}
\frac{a}{b} &< \sqrt{c} \\
\Rightarrow \frac{a+b}{b} &< \frac{\sqrt{c}b+b}{b} \\
\Rightarrow \frac{b}{a+b} &> \frac{b}{\sqrt{c}b+b} \\
\Rightarrow \frac{b}{a+b} &> \left(\frac{\sqrt{c}b-b}{\sqrt{c}b-b}\right)\left(\frac{b}{\sqrt{c}b+b}\right) \\
&= \frac{\sqrt{c}-1}{c-1} \\
\Rightarrow \frac{bc-b}{a+b} &> \sqrt{c}-1 \\
\Rightarrow \frac{bc-b}{a+b}+\frac{a+b}{a+b} &> \sqrt{c} \\
\Rightarrow \frac{a+bc}{a+b} &> \sqrt{c}
\end{aligned}
$$

Problem 6

A standard domino is a rectangular tile with a line dividing the rectangular face into two equal-sized squares. Each square is decorated with a number of pips (including zero pips – a blank end). A Double N set of dominos is composed of one each of all possibilities with the number of pips on one square end less than or equal to the number of pips on the other square end and the maximum number of pips on any square end equal to N. The most common set of dominos is called the Double Six set.

1. It is well known that there are 28 dominos in a Double Six set. Show that the number of dominos in a Double 48 set is a perfect square.
2. Prove that it is not possible to tile any square region without gaps or overlaps using a complete Double N set of dominos.
3. A tri-omino is an equilateral triangular tile with lines dividing the triangular face into four equilateral triangles decorated with blanks or pips. There are $\frac{1}{6}(N+3)(N+2)(N+1)$ tri-ominos in a Triple N set. Is it possible to tile a triangular region without gaps or overlaps using a complete Triple N set of tri-ominos if $N > 1$?

Solution 6

1. The number of dominos in a Double Six set is

$$(6+1)+(5+1)+(4+1)+\ldots 1 = 28.$$

The number of dominos in a Double N set is

$$\begin{aligned}\sum_{k=0}^{N}(k+1) &= N+1+\frac{1}{2}N(N+1)\\ &= \frac{1}{2}(N+2)(N+1)\end{aligned}$$

Note that if $N = 48$ then

$$\frac{1}{2}(N+2)(N+1) = \frac{1}{2}(50)(49) = (5^2)(7^2) = (5\times 7)^2$$

2. Let a denote the length of the short side of a domino. The total area of all faces in a Double N set is given by

$$\frac{1}{2}(N+1)(N+2)(2a)a = (N+1)(N+2)a^2.$$

Without loss of generality suppose that along one side of the square region there are m dominos with long side $2a$ and n dominos with short side a where m and n are integers. The total area of the square region is then

$$(m(2a)+n(a))^2 = (2m+n)^2a^2 = L^2a^2$$

where $L = 2m+n$ is an integer. Thus the square region can only be tiled without overlaps and gaps if there exists an integer L for which

$$L^2 = (N+1)(N+2)$$

but the product of two consecutive integers cannot be a perfect square. (This is easy to see by inspection since there are no integers L for which $(N+1) < L < (N+2)$).

3. It is easy to see that M tri-ominos can tile a triangular region without gaps or overlaps if M is a perfect square. It is easy to verify that $M^2 = \frac{1}{6}(N+3)(N+2)(N+1)$ has the trivial solution $N = 1, M = 2$. More generally note that in part (i) we have shown that

$$\frac{1}{2}(48+1)(48+2) = 35^2$$

or equivalently

$$\frac{1}{2}(47+2)(47+3) = 35^2.$$

But also note that

$$\frac{1}{3}(47+1) = 4^2$$

and then

$$\frac{1}{6}(47+1)(47+2)(47+3) = 4^2\times 35^2 = 140^2,$$

so that it is possible to tile a triangular region without gaps or overlaps with a Triple 47 set of $140^2 = 19,600$ tri-ominos. Indeed this is the only Triple N set with $N > 1$ that can tile a triangular region, but the proof is non-trivial.

Senior Division – Problems and Solutions

Problem 1

A standard domino is a rectangular tile with a line dividing the rectangular face into two equal-sized squares. Each square is decorated with a number of pips (including zero pips – a blank end). A Double N set of dominos is composed of one each of all possibilities with the number of pips on one square end less than or equal to the number of pips on the other square end and the maximum number of pips on any square end equal to N. The most common set of dominos is called the Double Six set.

1. It is well known that there are 28 dominos in a Double Six set. Show that the number of dominos in a Double 48 set is a perfect square.

2. Prove that it is not possible to tile any square region without gaps or overlaps using a complete Double N set of dominos.

3. A tri-omino is an equilateral triangular tile with lines dividing the triangular face into four equilateral triangles decorated with blanks or pips. There are $\frac{1}{6}(N+3)(N+2)(N+1)$ tri-ominos in a Triple N set. Is it possible to tile a triangular region without gaps or overlaps using a complete Triple N set of tri-ominos if $N > 1$?

Solution 1

1. The number of dominos in a Double Six set is

$$(6+1)+(5+1)+(4+1)+\ldots 1 = 28.$$

   The number of dominos in a Double N set is

$$\begin{aligned}\sum_{k=0}^{N}(k+1) &= N+1+\frac{1}{2}N(N+1) \\ &= \frac{1}{2}(N+2)(N+1)\end{aligned}$$

   Note that if $N = 48$ then

$$\frac{1}{2}(N+2)(N+1) = \frac{1}{2}(50)(49) = (5^2)(7^2) = (5\times 7)^2$$

2. Let a denote the length of the short side of a domino. The total area of all faces in a Double N set is given by

$$\frac{1}{2}(N+1)(N+2)(2a)a = (N+1)(N+2)a^2.$$

Without loss of generality suppose that along one side of the square region there are m dominos with long side $2a$ and n dominos with short side a where m and n are integers. The total area of the square region is then

$$(m(2a) + n(a))^2 = (2m + n)^2 a^2 = L^2 a^2$$

where $L = 2m + n$ is an integer. Thus the square region can only be tiled without overlaps and gaps if there exists an integer L for which

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But also note that

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and then

$$\frac{1}{6}(47 + 1)(47 + 2)(47 + 3) = 4^2 \times 35^2 = 140^2,$$

so that it is possible to tile a triangular region without gaps or overlaps with a Triple 47 set of $140^2 = 19,600$ tri-ominos. Indeed this is the only Triple N set with $N > 1$ that can tile a triangular region, but the proof is non-trivial.

Problem 2

In Australian elections, the two-party preferred vote assigns a vote to either the Australian Labor Party (ALP) or the Liberal National Coalition (Coalition). A polling company notices the following trend in the run up to an election. On a two-party preferred basis, in any one month period, 30% of the leading party's vote shifts to the opposing party and 20% of the opposing party's vote shifts to the leading party. Five months before the election the ALP has 55% of the total vote and the Coalition has 45% of the vote on a two-party preferred basis.

1. If the trend continues who would win the election and what percentage of the vote would they have?

2. Assuming the trend continues after the election what is the maximum vote, to the nearest percent, that the losing party could attain in the next four year term of government?

Solution 2

Let $A(n)$ denote the proportion of the vote for the ALP at time n and let $B(n)$ denote the proportion of the vote for the Coalition at time n with one month corresponding to the unit of time.

The vote for the ALP in month $n+1$ will be

$$A(n+1) = \begin{cases} A(n) - \frac{30}{100}A(n) + \frac{20}{100}B(n), & A(n) > B(n) \\ \\ A(n) - \frac{20}{100}A(n) + \frac{30}{100}B(n), & A(n) < B(n) \end{cases}$$

The vote for the Coalition in month $n+1$ will be

$$B(n+1) = \begin{cases} B(n) - \frac{30}{100}B(n) + \frac{20}{100}A(n), & A(n) < B(n) \\ \\ B(n) - \frac{20}{100}B(n) + \frac{30}{100}A(n), & B(n) < A(n) \end{cases}$$

At all times we have $A(n)+B(n) = 1$ so that it is sufficient to consider the dynamics of the vote for one party, the ALP say, in which case the governing equation simplifies to

$$A(n+1) = \begin{cases} \frac{20}{100} + \frac{50}{100}A(n), & A(n) > \frac{50}{100} \\ \\ \frac{30}{100} + \frac{50}{100}A(n), & A(n) < \frac{50}{100} \end{cases}$$

1. It is a simple matter to start with $A(0) = 0.55$ and then calculate $A(1) = 0.475$, $A(2) = 0.5375$, $A(3) = 0.46875$, $A(4) = 0.534375$, $A(5) = 0.4671875$ so that the ALP would have less than half the vote at the time of the election.

2. Note that the proportion of the vote to the ALP quickly becomes cyclic so that if $A(k) < 0.5$ then $A(k+1) > 0.5$ and vice versa. Thus we can write

$$A(k+2) = \frac{20}{100} + \frac{50}{100}A(k+1) = \frac{20}{100} + \frac{50}{100}\left(\frac{30}{100} + \frac{50}{100}A(k)\right).$$

   After a long time we expect $A(k+2) = A(k) = X$ with solution $X = \dfrac{7}{15}$. Thus after long times we anticipate the vote for any one party will alternate from one month to the next between $X = \dfrac{7}{15}$ and $1 - X = \dfrac{8}{15}$. The maximum vote for the losing party if the trend continues is thus $\dfrac{8}{15}$ or 53.3% of the vote.

Problem 3

Two friends pass time playing a simple game with standard dice; small cubes with the faces displaying pips that number from one to six.

They start by rolling a single die in turns until one of the friends rolls a six. They then roll a pair of dice in turns until the total on the faces is seven. The first to roll seven is declared the winner.

1. What is the probability that the person who rolled the single die first will roll the first six?
2. If the first person to roll a six has the first turn in rolling the pair of dice what is the probability that the person who rolled the single die first will win the game?

Solution 3

1. The probability that the first player rolls the first six on their first roll is $\frac{1}{6}$, the probability they do not roll a six on the first roll (and nor does their opponent) but they do on their second roll is $\left(\frac{5}{6}\right)^2 \frac{1}{6}$, the probability that they do not roll a six on their first n rolls (and nor does their opponent) but they do on the $(n+1)$th roll is $\left(\frac{5}{6}\right)^{2n} \frac{1}{6}$. The probability that the first player rolls the first six is thus

   $$p = \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^{2n} \frac{1}{6}.$$

   Recall the geometric series

   $$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

   and then the player who rolled first has the probability of

   $$p = \left(\frac{1}{1-\left(\frac{5}{6}\right)^2}\right) \frac{1}{6} = \frac{6}{11}.$$

   As an aside, the probability that the player who rolls second is the player who rolls the first six is thus

   $$q = \frac{5}{11}.$$

2. There are two ways that the person who rolls the first single die can win the game: (i) if they roll the first six and then roll the first seven, or (ii) they do not roll the first six but they roll the first seven.
   Now consider the probability that the first player to roll a pair is the first player to roll a seven. With a pair of dice there are six ways for the faces to add to seven; a one and a six (or a six and a one), a two and a five (or a five and a two), a three and a four (or a four and a three), out of six times six possible outcomes. The probability of rolling a seven is thus $= \frac{6}{36} = \frac{1}{6}$ and the probability of not rolling

a seven is $= \frac{5}{6}$. Thus similar to above the probability that the first player to roll a pair is the first player to roll a seven is $\frac{6}{11}$ and the probability that the second player to roll a pair is the first to roll a seven is $\frac{5}{11}$.

The probability that the person who rolled the single die first will win the game is thus the probability that they rolled the first six and then (being the first to roll a pair) they rolled the first seven plus the probability that they did not roll the first six and then (being the second to roll a pair) they rolled the first seven:

$$P = \frac{6}{11} \times \frac{6}{11} + \frac{5}{11} \times \frac{5}{11} = \frac{36+25}{121} = \frac{61}{121}.$$

Problem 4

Find the product of all distinct numbers of the form $x^{\frac{1}{x}}$ where $x = 2^k$ and k is a non-negative integer.

Solution 4

Consider

$$a_k = (2^k)^{\frac{1}{2^k}} = 2^{\frac{k}{2^k}}$$

then we wish to find the product of all distinct a_k where k is a non-negative integer. Note that $a_0 = 1, a_1 = \sqrt{2}, a_2 = \sqrt{2}$ and $a_{k+1} < a_k$ for all $k > 1$ so that the desired result is

$$\begin{aligned} P &= \frac{1}{\sqrt{2}} \prod_{k=0}^{\infty} a_k \\ &= \frac{1}{\sqrt{2}} \prod_{k=0}^{\infty} 2^{\frac{k}{2^k}} \\ &= \frac{1}{\sqrt{2}} 2^{\sum_{k=0}^{\infty} \frac{k}{2^k}} \end{aligned}$$

We now consider the sum

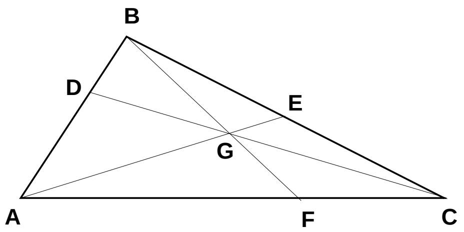
$$\begin{aligned} S &= \sum_{k=0}^{\infty} \frac{k}{2^k} \\ &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1+1}{2} + \frac{1+2}{4} + \frac{1+3}{8} + \frac{1+4}{16} + \cdots\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) + \frac{1}{2}S \end{aligned}$$

Hence
$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$
But this is the standard geometric series
$$S = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2.$$

We now have
$$P = \frac{1}{\sqrt{2}} 2^S = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

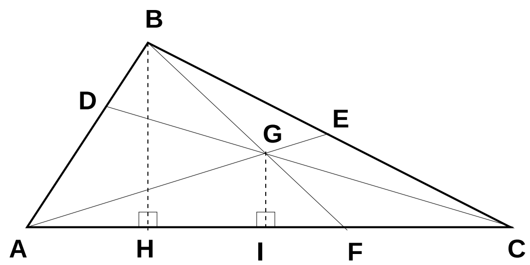
Problem 5

Consider the triangle shown below with vertices A, B, C where point D lies on the side AB, point E lies on the side BC and point F lies on the side AC, and the three lines AE, BF and CD intersect at a common point G.



Show that
$$\frac{\text{Area}(\triangle CGF)}{\text{Area}(\triangle AGF)} = \frac{\text{Area}(\triangle BGC)}{\text{Area}(\triangle BGA)}$$

Solution 5



From the altitudes GI and BH shown on the figure we first deduce

$$\begin{aligned}
\text{Area}(\triangle AGF) &= \frac{1}{2}|AF||GI| \\
\text{Area}(\triangle CGF) &= \frac{1}{2}|CF||GI| \\
\text{Area}(\triangle ABF) &= \frac{1}{2}|AF||BH| \\
\text{Area}(\triangle CBF) &= \frac{1}{2}|CF||BH|
\end{aligned}$$

Thus

$$\frac{CF}{AF} = \frac{\text{Area}(\triangle CGF)}{\text{Area}(\triangle AGF)} = \frac{\text{Area}(\triangle CBF)}{\text{Area}(\triangle ABF)}$$

But

$$\frac{\text{Area}(\triangle CBF)}{\text{Area}(\triangle ABF)} = \frac{\text{Area}(\triangle CGF) + \text{Area}(\triangle BGC)}{\text{Area}(\triangle AGF) + \text{Area}(\triangle BGA)}$$

and hence

$$\frac{\text{Area}(\triangle CGF)}{\text{Area}(\triangle AGF)} = \frac{\text{Area}(\triangle CGF) + \text{Area}(\triangle BGC)}{\text{Area}(\triangle AGF) + \text{Area}(\triangle BGA)}.$$

The result

$$\frac{\text{Area}(\triangle CGF)}{\text{Area}(\triangle AGF)} = \frac{\text{Area}(\triangle BGC)}{\text{Area}(\triangle BGA)}$$

now immediately follows.

Problem 6

Prove that

$$\begin{aligned} &2 \times 2011 \times (2011^3 + 1)(2010^3 + 1) \cdots (2^3 + 1) > \\ &3 \times 2010 \times (2011^3 - 1)(2010^3 - 1) \cdots (2^3 - 1) \end{aligned}$$

Solution 6

The inequality can be re-written as

$$\frac{2 \times 2011}{3 \times 2010} > \prod_{j=2}^{2011} \frac{j^3 - 1}{j^3 + 1}.$$

For abbreviation, we define

$$T(j) = j^2 + j + 1,$$

and we see note that

$$T(j-1) = j^2 - j + 1.$$

The right-hand side of the above inequality then reads

$$\begin{aligned}
\prod_{j=2}^{2011} \frac{j^3-1}{j^3+1} &= \prod_{j=1}^{2011} \frac{(j-1)T(j)}{(j+1)T(j-1)} \\
&= \frac{2010!}{2012!} \times \frac{2T(2011)}{T(1)} \\
&= \frac{2}{3} \times \frac{2011^2+2011+1}{2011 \times 2012} \\
&= \frac{2}{3} \times \frac{2011 \times 2012 + 1}{2011 \times 2012} \\
&= \frac{2}{3} \times \left(1 + \frac{1}{2011 \times 2012}\right) \\
&< \frac{2}{3} \times \left(1 + \frac{1}{2010}\right) \\
&= \frac{2}{3} \times \frac{2011}{2010}.
\end{aligned}$$