Parabola Volume 48, Issue 3 (2012)

2012 University of New South Wales School Mathematics Competition [1]

Junior Division – Problems and Solutions

Problem 1
The infinite nested radical

$$c = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{\dots}}}}}$$

converges. Find c.

Solution 1
Note that if

$$c = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{\dots}}}}}$$

then

$$c = \sqrt{1 + 2c}.$$

Square both sides and then

$$c^2 = 1 + 2c$$

which has solutions $1 + \sqrt{2}$ and $1 - \sqrt{2}$. Clearly we are seeking a positive answer so that $c = 1 + \sqrt{2}$.

Problem 2
In how many ways can 10000 be written as a sum of consecutive odd positive integers?

Solution 2
Suppose that k is odd then

$$\begin{aligned} k + (k+2) + (k+2(2)) + \cdots (k + 2(n-1)) &= 10000 \\ nk + 2(1 + \cdots + (n-1)) &= 10000 \\ nk + n(n-1) &= 10000 \\ n(k + (n-1)) &= 10000 \end{aligned}$$

[1] The problems and solutions were compiled, created, refined with contributions from David Angell, Chris Angstmann, Peter Brown, Michael Cowling, David Crocker, Bruce Henry (Director), David Hunt, Tyrone Liang, Adrian Miranda.

If n is odd then $k+(n-1)$ is also odd and then $n(k+(n-1))$ must also be odd but this contradicts the equality with 10000. Thus n must be even. Thus a solution is obtained whenever $10000 = 2^4 \times 5^4$ is the product of two even integers.

There are eight possibilities, $n = 2, 4, 8, 10, 20, 40, 50, 100$.

Problem 3

Consider the following array of integers

Row 0					1				
Row 1				1	1	1			
Row 2			1	2	3	2	1		
Row 3		1	3	6	7	6	3	1	
Row 4	1	4	10	16	19	16	10	4	1

in which every number is the sum of the number n directly above and the numbers one to the left and one to the right of n. A blank space indicates the number zero. Thus $16 = 3 + 6 + 7$.

1. Prove that the sum of entries in row k is 3^k.
2. Prove that there is at least one even number in each row beyond row 1.
3. Prove that the third (non-zero) number from the left in row k is given by $\frac{1}{2}k(k+1)$.

Solution 3

We will use the notation $b_{k,j}$ to denote the jth non-zero entry from the left in the kth row. Note that row k is composed of $2k+1$ non-zero numbers and $b_{k,1} = 1, b_{k,2} = k$ for all $k \geq 1$.

1. The statement is true for $k = 0$ and $k = 1$. Now suppose that for some row k we have
$$1 + b_{k,2} + b_{k,3} + \ldots + 1 = 3^k.$$
Note that in the $k+1$th row $b_{k+1,j+1} = b_{k,j-1} + b_{k,j} + b_{k,j+1}$. Then
$$\begin{aligned} &1 + b_{k+1,2} + b_{k+1,3} + b_{k+1,4} + \cdots + 1 \\ &= 1 + (1 + b_{k,2}) + (1 + b_{k,2} + b_{k,3}) + (b_{k,2} + b_{k,3} + b_{k,4}) + \cdots + 1 \\ &= 3(1 + b_{k,2} + b_{k,3} \cdots + 1) \\ &= 3^{k+1}. \end{aligned}$$
Thus by induction the statement holds for all rows.
2. Clearly every even numbered row k contains an even number since $b_{k,2} = k$.

More generally it is useful to consider the pattern of numbers in mod 2.

Row 0							1						
Row 1						1	1	1					
Row 2					**1**	**0**	**1**	**0**	1				
Row 3				**1**	**1**	**0**	**1**	0	1	1			
Row 4			**1**	**0**	**0**	**0**	1	0	0	0	1		
Row 5		**1**	**1**	**1**	**0**	1	1	1	0	1	1	1	
Row 6	**1**	**0**	**1**	**0**	0	0	1	0	0	0	1	0	1

Looking at the first four entries in each row, starting at Row 2, we see that the pattern repeats by construction, after Row 6, with a 0 in each row.

3. Suppose that $b_{k,3} = \frac{1}{2}k(k+1)$ then

$$\begin{aligned} b_{k+1,3} &= b_{k,1} + b_{k,2} + b_{k,3} \\ &= 1 + k + \frac{1}{2}k(k+1) \\ &= (1+k)(1+\frac{k}{2}) \\ &= \frac{1}{2}((k+1)(k+2)) \end{aligned}$$

and the result holds for all rows by induction.

Problem 4

Given any two people we may classify them as friends, enemies or strangers. Prove that in a gathering of seventeen people there must be either three mutual friends or three mutual enemies or three mutual strangers. (This problem appeared in *Parabola* Vol 15, Problem 439 (1979))

Solution 4

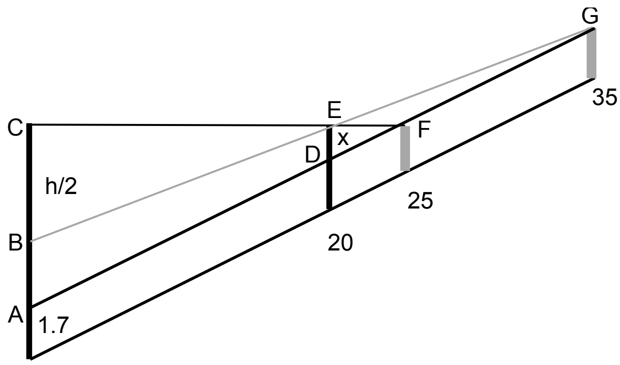
Let A be one of the 17 people and partition the remaining 16 into 3 sets: E the enemies of A, F the friends of A and U those unacquainted with A. The largest of these 3 sets must contain at least 6 people. Suppose that F contains six or more people. If any 2 members of F, say B and C, are friends, then A, B, C are mutual friends. Now suppose no 2 people in F are friends. Choose any B in F and partition the remaining 5 people in F into 2 sets: X the enemies of B and Y those unacquainted with B. Either X or Y contains at least 3 people. If it is X and any 2 people in X are enemies, then B, C, D are 3 mutual enemies. Otherwise, no 2 people in X are enemies and then all the people in X are mutual strangers. Similarly if Y contains 3 or more people, we obtain either 3 mutual strangers (B and 2 people in Y), or 3 mutual enemies, all in Y. The same argument can be repeated if either E or U has 6 members.

Problem 5

A person of height 1.7 metres leaves a tall building at ground level and walks in a straight line direction up a path of constant gradient. They walk under a tall billboard after twenty metres and continue walking up the path for another five metres at which point they turn around and notice that the top of the billboard aligns horizontally with the top of the building. They continue along up the path a further ten metres where they turn around again and notice that the top half of the building is now visible above the billboard. The height of the building is much greater than the height of the billboard which is much greater than the height of the person. What is the height of the building?

Solution 5

In the figure shown $\triangle ACF$ is similar to $\triangle DEF$ and $\triangle ABG$ is similar to $\triangle DEG$ from



which it follows that

$$\frac{h-1.7}{25} = \frac{x}{5}$$

and

$$\frac{h-1.7}{35} = \frac{x}{15}$$

respectively. Eliminating x in these two equations we obtain the result $h = 27.2$ metres.

Problem 6

A travelling sales person tours towns A, B, C, D, E and stays overnight in one of the towns. If they stay overnight in town A then the next night they stay in town B. If they stay overnight in town B then the next night they stay in town C. If they stay overnight in town C then the next night they stay in town D. If they stay overnight in town D then the next night they stay in town E. If they stay overnight in town E they roll two fair dice to determine whether they will return to D for the next night or move on to town A for the next night. They then continue their tour either from D to E or from A to B, etc. What is the long-term probability of finding them in town E on any given night in each of the scenarios below:

Scenario 1

They return from E to D if the roll of the dice adds up to a number divisible by two, otherwise they move on from E to A.

Scenario 2

They return from E to D if the roll of the dice adds up to a number divisible by three, otherwise they move on from E to A.

Solution 6

Let P_X denote the long-term probability that the sales person stays overnight in town X and let p_{YZ} denote the transition probability that the sales person goes from Y to Z. Then $p_{AB} = 1, p_{BC} = 1, p_{CD} = 1, p_{DE} = 1, p_{ED} = p, p_{EA} = 1 - p$ where $0 \leq p \leq 1$.

We also have $P_A = P_E \times p_{EA}$, $P_B = P_A \times p_{AB}$, $P_C = P_B \times p_{BC}$, $P_D = P_C \times p_{CD} + P_E \times p_{ED}$, $P_E = P_D \times p_{DE}$. Thus $P_A = P_E \times (1 - p)$, $P_B = P_A$, $P_C = P_A$, $P_D = P_A + P_E \times p$, $P_E = P_D$. But $P_A + P_B + P_C + P_D + P_E = 1$ so that $4P_A + (p+1)P_E = 1$. Finally eliminating P_A from $P_A = P_E \times (1-p)$ and $4P_A + (p+1)P_E = 1$, we have $4P_E \times (1-p) + (p+1)P_E = 1$ and then $P_E = \frac{1}{5-3p}$.

In Scenario 1 the sum of the dice is divisible by 2 if the sum is one of $2, 4, 6, 8, 10, 12$ so that $p = \frac{1+3+5+5+3+1}{36} = \frac{1}{2}$ and $P_E = \frac{2}{7}$.

In Scenario 2 the sum of the dice is divisible by 3 if the sum is one of $3, 6, 9, 12$ so that the probability is $p = \frac{2+5+4+1}{36} = \frac{1}{3}$ and $P_E = \frac{1}{4}$.

Senior Division – Problems and Solutions

Problem 1

A travelling sales person tours towns A, B, C, D, E and stays overnight in one of the towns. If they stay overnight in town A then the next night they stay in town B. If they stay overnight in town B then the next night they stay in town C. If they stay overnight in town C then the next night they stay in town D. If they stay overnight in town D then the next night they stay in town E. If they stay overnight in town E they roll two fair dice to determine whether they will return to D for the next night or move on to town A for the next night. They then continue their tour either from D to E or from A to B, etc. What is the long-term probability of finding them in town E on any given night in each of the scenarios below:

Scenario 1

They return from E to D if the roll of the dice adds up to a number divisible by two, otherwise they move on from E to A.

Scenario 2

They return from E to D if the roll of the dice adds up to a number divisible by three, otherwise they move on from E to A.

Solution 1

See Solution 6 in the Junior Division.

Problem 2

Show that the infinite product

$$\left(\frac{1}{n}\right)^{\frac{1}{n}} \times \left(\frac{1}{n}\right)^{\frac{2}{n^2}} \times \left(\frac{1}{n}\right)^{\frac{3}{n^3}} \times \cdots \times \left(\frac{1}{n}\right)^{\frac{j}{n^j}} \times \cdots = \left(\frac{1}{n}\right)^{\frac{n}{(n-1)^2}}$$

Solution 2

Solution 2.1

$$\left(\frac{1}{n}\right)^{\frac{1}{n}} \times \left(\frac{1}{n}\right)^{\frac{2}{n^2}} \times \left(\frac{1}{n}\right)^{\frac{3}{n^3}} \times \cdots \times \left(\frac{1}{n}\right)^{\frac{j}{n^j}} \times \cdots = \left(\frac{1}{n}\right)^{\sum_{j=0}^{\infty} \frac{j}{n^j}}$$

Now consider

$$\begin{aligned}
S &= \sum_{j=0}^{\infty} \frac{j}{n^j} \\
&= \frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \frac{5}{n^5} + \cdots \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{2}{n} + \frac{3}{n^2} + \frac{4}{n^3} + \frac{5}{n^4} + \cdots\right) \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{1+1}{n} + \frac{1+2}{n^2} + \frac{1+3}{n^3} + \frac{1+4}{n^4} + \cdots\right) \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots\right) + \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \cdots\right) \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots\right) + \frac{1}{n}S
\end{aligned}$$

Using the well-known result for the geometric series

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots = \frac{1}{n-1}$$

we now have

$$S = \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n-1}\right) + \frac{1}{n}S$$

and then it is a simple matter to solve for

$$S = \frac{n}{(n-1)^2}$$

Solution 2.2
Suppose $-1 < x < 1$ then

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

If we treat x as a variable and differentiate with respect to x then

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$$

and

$$x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}.$$

If we now substitute $x = \frac{1}{n}$ into the above we obtain the result

$$\frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \cdots = \frac{\frac{1}{n}}{(1-\frac{1}{n})^2} = \frac{n}{(n-1)^2},$$

as required.

Problem 3
Provide an example of a function $f(x)$ whose inverse $f^{-1}(x)$ is also its derivative $f'(x)$.

Solution 3
It is natural to attempt trial solutions. Consider a power law $f(x) = ax^b$ then $f^{-1}(x) = \left(\frac{x}{a}\right)^{\frac{1}{b}}$ and $f'(x) = bax^{b-1}$ are both power laws and they are equal if the powers are equal and the coefficients are equal. The powers are equal if $\frac{1}{b} = b - 1$ so that $b^2 - b - 1 = 0$ and $b = \frac{1 \pm \sqrt{5}}{2}$. The coefficients agree if $\left(\frac{1}{a}\right)^{\frac{1}{b}} = ba$. Solving for a we obtain

$$a = \left(\frac{1}{b}\right)^{\frac{b}{b+1}}.$$

Thus if we consider $b = \frac{1+\sqrt{5}}{2}$ then

$$a = \left(\frac{2}{1+\sqrt{5}}\right)^{\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right)}$$

and

$$f(x) = \left(\frac{2}{1+\sqrt{5}}\right)^{\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right)} x^{\frac{1+\sqrt{5}}{2}}.$$

This can be written more simply as

$$f(x) = \left(\frac{1+\sqrt{5}}{2}\right)^{\left(\frac{1-\sqrt{5}}{2}\right)} x^{\frac{1+\sqrt{5}}{2}}$$

Problem 4
A coin is selected at random from a bag containing two coins. One of the coins is an unbiased coin with a head and a tail and the other coin has two heads. The selected coin is tossed in the air and lands heads up three times in succession.

1. What is the probability that the coin that was thrown has two heads?
2. What is the probability that the next three throws of this coin will be heads up?

Solution 4

1. Consider selecting a coin and throwing it three times. This gives $2^3 = 16$ possible outcomes:
   $HHH, THH, HHT, HTH, \ldots, TTH, TTT$.
   Exactly nine of them are HHH, eight come from the double headed coin, the other from the fair coin. So given HHH has occurred $p(\text{fair coin}) = \frac{1}{9}$ and $p(\text{two headed coin}) = \frac{8}{9}$.

2. $p(HHH) = \frac{8}{9} \times 1 + \frac{1}{9} \times \frac{1}{8} = \frac{65}{72}$.

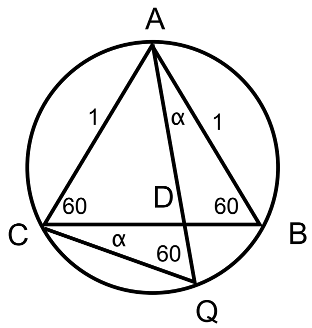
Problem 5

A line drawn from the vertex A of the equilateral triangle ABC meets the side BC at D and the circumcircle of the triangle at point Q. Prove that

$$\frac{1}{QD} = \frac{1}{QB} + \frac{1}{QC}.$$

Solution 5

From the figure shown it can be seen that $ABQC$ is a special quadrilateral.



Let $AB = BC = CA = 1$. Since $\triangle ABC$ is equilateral $\angle ABC = \angle CAB = \angle ACB = 60°$. Using angles in the same segment $\angle AQC = \angle AQB = 60°$.

Using areas it can be seen that

$$\text{area} \quad \triangle CQB = \text{area} \quad \triangle CQD + \text{area} \quad \triangle DQB.$$

Thus

$$\frac{1}{2}QC.QB.\sin 120° = \frac{1}{2}QC.QD\sin 60° + \frac{1}{2}QD.QB.\sin 60°.$$

But $\sin 120° = \sin 60°$ so that

$$QB.QC = QD.QC + QD.QB$$

and then dividing throughout by $QB.QC.QD$ we obtain

$$\frac{1}{QD} = \frac{1}{QB} + \frac{1}{QC}$$

as required.

Problem 6

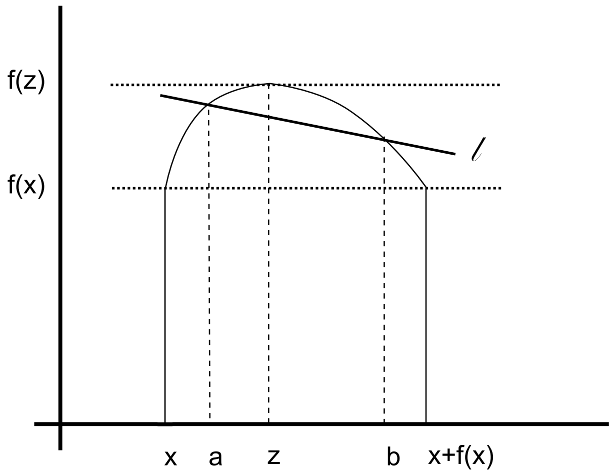
Let $f(x)$ denote a strictly positive continuous function defined on all real numbers with the property that $f(2012) = 2012$ and $f(x) = f(x + f(x))$ for all x. Prove that $f(x) = 2012$ for all x.

Solution 6

We are given $f(x) = f(x + f(x))$ for all x so that if we replace x by $x + f(x)$ we have $f(x + f(x)) = f(x + f(x) + f(x + f(x))$ and if we now use the equality $f(x + f(x)) = f(x)$ we obtain $f(x) = f(x + 2f(x))$. Continuing in this fashion we have $f(x) = f(x + nf(x))$ for all x and all integers n.

We consider a proof by contradiction to show that $f(x)$ is a constant function. Suppose that $f(x)$ is not a constant function. Without loss of generality we may assume there exists $z \in (x, x + f(x))$ such that $f(x) < f(z) < 2f(x)$, and furthermore $f(z) = f(z + nf(z))$ for all integers n.

Clearly there exists a straight line ℓ that separates the point $(z, f(z))$ on the graph from points $(x, f(x))$ and $(x + f(x), f(x))$. Without loss of generality we suppose that the straight line ℓ is given by $y = -\frac{1}{m}x + c$ where m is a positive integer. It follows from the continuity of $f(x)$ that there are at least two points $(a, f(a))$ and $(b, f(b))$ with $a \neq b$ that lie on the graph and the straight line. This is shown schematically in the figure.



Thus we have $c = a + mf(a)$ and $c = b + mf(b)$ so that $f(c) = f(a + mf(a))$ and $f(c) = f(b + mf(b))$. But $f(a + mf(a)) = f(a)$ and $f(b + mf(b)) = f(b)$ so that $f(c) = f(a) = f(b)$. It further follows that $c = a + mf(c)$ and $c = b + mf(c)$ so that $a = b$, which is a contradiction.

Finally we have $f(x)$ is a constant function and we are given $f(2012) = 2012$ so that $f(x) = 2012$ for all x.