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2014 UNSW School Mathematics Competition Junior Division - Problems and Solutions

Solutions by David Crocker, UNSW, Australia.

Problem 1
Find

$$S = 1 + 11 + 111 + \cdots + \underbrace{11\ldots1}_{2014 \text{ digits}}.$$

Proof. [Solution] By sum of the geometric progression formula:

$$\begin{aligned}\underbrace{11\ldots1}_{k \text{ digits}} &= 1 + 10 + 10^2 + \cdots + 10^{k-1} \\ &= \frac{10^k - 1}{10 - 1} \\ &= \frac{1}{9}(10^k - 1),\end{aligned}$$

so we compute

$$\begin{aligned} S &= \sum_{k=1}^{2014} \frac{1}{9}(10^k - 1) \\ &= \frac{1}{9}\left[\sum_{k=1}^{2014} 10^k - \sum_{k=1}^{2014} 1\right] \\ &= \frac{1}{9}\left[10(1 + 10 + \cdots + 10^{2013}) - 2014\right] \\ &= \frac{1}{9}\left[\frac{10(10^{2014} - 1)}{9} - 2014\right] \quad \text{(sum of a GP)} \\ &= \frac{1}{81}\left[10^{2015} - 10 - 9 \cdot 2014\right] \\ &= \frac{1}{81}\left[10^{2015} - 18136\right] \end{aligned}$$

We can find the decimal representation of S as follows. By setting

$$x = 10^{2015} - 18136,$$

we see that

$$S = \frac{x}{81}.$$

On the other hand, writing out x in base 10 we find,

$$x = \underbrace{99\ldots9}_{2010\text{ digits}} \Big| 81864.$$

Hence,

$$\frac{x}{9} = \underbrace{11\ldots1}_{2010\text{ digits}} \Big| 09096.$$

We now divide by 9 again. Since

$$\underbrace{11\ldots1}_{9\text{ digits}} = 9 \times (012345679)$$

then extracting groups of 9 1s in $x/9$, and dividing by 9 and as

$$2010 = 9 \times 223 + 3$$

then $S = x/81$ in base 10 consists of 223 groups of 012345679 followed by

$$11109096 \div 9 = 01234344.$$

Hence,

$$\begin{aligned} S &= \underbrace{(012345679)(012345679)\cdots(012345679)}_{223\text{ times}} \Big| 01234344 \\ &= \underbrace{(123456790)(123456790)\cdots(123456790)}_{223\text{ times}} \Big| 1234344. \end{aligned}$$

□

Problem 2
Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 1$. Prove that

$$a_1^2 + \cdots + a_n^2 \geq \frac{1}{n}.$$

Proof. [Solution 1] A possible solution can be

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} \left(a_i - \frac{1}{n}\right)^2 \\ &= \sum_{i=1}^{n} \left(a_i^2 - \frac{2a_i}{n} + \frac{1}{n^2}\right) \\ &= \sum_{i=1}^{n} a_i^2 - \frac{2}{n}\sum_{i=1}^{n} a_i + \frac{1}{n^2}\sum_{i=1}^{n} 1 \\ &= \sum_{i=1}^{n} a_i^2 - \frac{2}{n}\cdot 1 + \frac{1}{n^2}\cdot n \\ &= \sum_{i=1}^{n} a_i^2 - \frac{2}{n} + \frac{1}{n} \\ &= \sum_{i=1}^{n} a_i^2 - \frac{1}{n}. \end{aligned}$$

Hence,

$$\frac{1}{n} \le \sum_{i=1}^{n} a_i^2.$$

□

Proof. [Solution 2] A similar solution is to shift

$$a_i = \frac{1}{n} + x_i \text{ for } 1 \le i \le n$$

and then compute

$$\begin{aligned} 1 &= \sum_{i=1}^{n} a_i \\ &= \sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} x_i \\ &= \frac{1}{n}\cdot n + \sum_{i=1}^{n} x_i \\ &= 1 + \sum_{i=1}^{n} x_i \end{aligned}$$

so

$$\sum_{i=1}^{n} x_i = 0.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n} a_i^2 &= \sum_{i=1}^{n} \left(\frac{1}{n} + x_i\right)^2 \\
&= \sum_{i=1}^{n} \left(\frac{1}{n^2} + \frac{2x_i}{n} + x_i^2\right) \\
&= \left(\sum_{i=1}^{n} \frac{1}{n^2}\right) + \frac{2}{n}\left(\sum_{i=1}^{n} x_i\right) + \left(\sum_{i=1}^{n} x_i^2\right) \\
&= n \cdot \frac{1}{n^2} + \frac{2}{n} \cdot 0 + \left(\sum_{i=1}^{n} x_i^2\right) \\
&= \frac{1}{n} + \left(\sum_{i=1}^{n} x_i^2\right) \\
&\geq \frac{1}{n}.
\end{aligned}$$

□

Proof. [Solution 3] Recall the Cauchy-Schwartz inequality:

$$\vec{a} \cdot \vec{b} \leq ||\vec{a}||\,||\vec{b}||,$$

where

$$\vec{a} = (a_1, a_2, \ldots, a_n) \text{ and } \vec{b} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{n} &= \frac{a_1 + a_2 + \cdots + a_n}{n} \\
&= \vec{a} \cdot \vec{b} \\
&\leq ||\vec{a}||\,||\vec{b}|| \\
&= \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} \frac{1}{n^2}} \\
&= \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2}
\end{aligned}$$

from which the result follows. □

Problem 3

Find all possible decimal digits you can use to fill places marked with an asterisk $*$, so that the following identity holds

$$*00*** = (***)^2.$$

Proof. [Solution] We want integers x with

$$100 \leq x \leq 999$$

such that for some integer a, $1 \leq a \leq 9$,

$$100000a \leq x^2 < 100000a + 1000$$
$$\text{i.e.} \quad 100\sqrt{10a} \leq x < 10\sqrt{1000a + 10}.$$

Rounding to 1 decimal place, a calculator gives the following results with the 7 solutions to x and x^2 in the last two columns:

a	$100\sqrt{10a}$	$10\sqrt{1000a+10}$	x	x^2
1	316.2	317.8	317	100489
2	447.2	448.3	448	200704
3	547.7	548.6	548	300304
4	632.4	633.2	633	400689
5	707.1	707.8	–	–
6	774.5	775.2	775	600625
7	836.6	837.2	837	700569
8	894.4	894.9	–	–
9	948.6	949.2	949	900601

□

Problem 4
Five speakers A, B, C, D and E take part in a conference. Find the total number of ways to organise the programme so that

a) A speaks immediately before B;

b) B does not speak before A.

Proof. [Solution] For a)

No. =	4	×	1	×	3!
	↓		↓		↓
	Choose a position for A		Place B in next position		Fill remaining positions

$= 4 \times 6 = 24$

For b)

$$\text{No.} = \sum_{j=1}^{4} \underbrace{1}_{\substack{\text{Place A in} \\ \text{position } j}} \times \underbrace{(5-j)}_{\substack{\text{Choose a later} \\ \text{position for B}}} \times \underbrace{3!}_{\substack{\text{Fill remain-} \\ \text{ing positions}}}$$

$$= (4+3+2+1) \times 6$$

$$= 10 \times 6 = 60$$

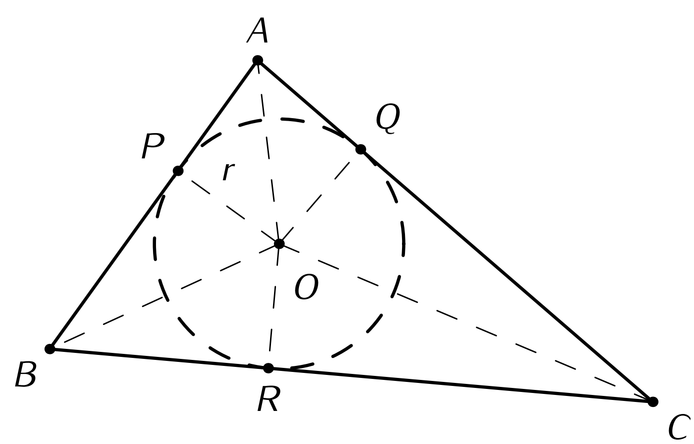
□

Problem 5

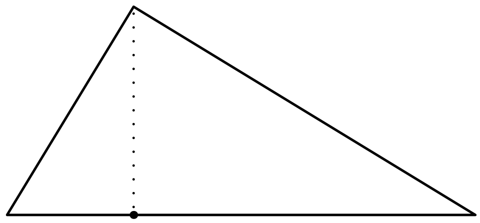
a) Prove that the radius of the inscribed circle to the triangle $\triangle ABC$ is given by

$$r = \frac{2S}{AB + BC + AC},$$

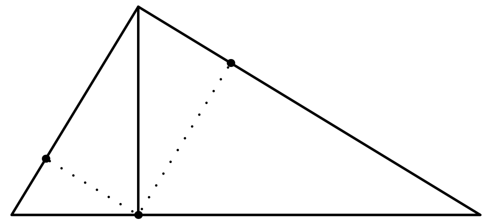
where S is the total area of the triangle $\triangle ABC$.



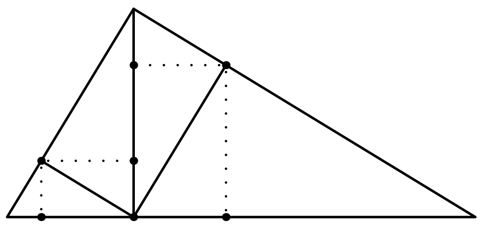
b) In a right-angled triangle, we draw the altitude onto the hypotenuse. This process is repeated in the two smaller right-angled triangles so formed and the process is then continued 2014 times, as shown in the diagram. A circle is inscribed in each of the resulting 2^{2014} triangles. Find the total area of these circles.



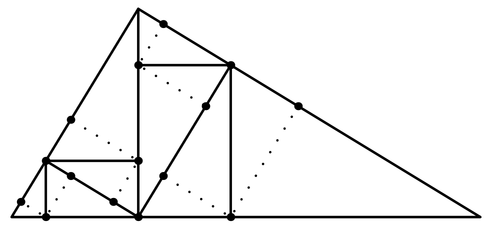
Stage 1



Stage 2



Stage 3



Stage 4

Proof. [Solution, part a] Let $AB = c,\ AC = b,\ BC = a$ (lengths). Let the incircle to $\triangle ABC$ meet the sides at P, Q, R as shown. Since the sides of $\triangle ABC$ are tangent to the in-circle, then PO, QO and RO are perpendicular to the respective sides AB, AC and BC. Hence PO is an altitude for $\triangle AOB$, QO is an altitude for $\triangle AOC$ and RO is an altitude for $\triangle BOC$, and these altitudes have length r, the radius of the incircle. Now $\triangle ABC$ is partitioned into sub-triangles $\triangle AOB,\ \triangle BOC,\ \triangle AOC$. Hence

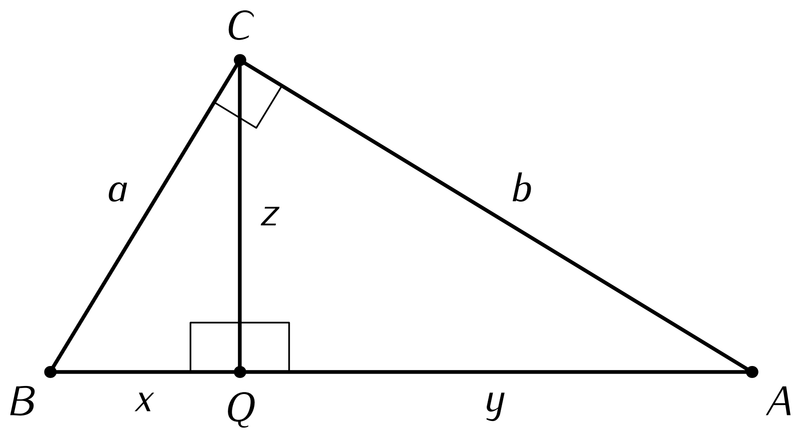
$$\begin{aligned} S = \text{Area}(\triangle ABC) &= \text{Area}(\triangle AOB) + \text{Area}(\triangle BOC) + \text{Area}(\triangle AOC) \\ &= \frac{1}{2}rc + \frac{1}{2}ra + \frac{1}{2}rb \\ &= \frac{r}{2}(c + b + a). \end{aligned}$$

Therefore,

$$r = \frac{2S}{a + b + c} = \frac{2S}{BC + AC + BC}.$$

□

Proof. [Solution, part b] Let $\triangle ABC$ be a right-angled triangle, w.l.o.g. let $\angle ACB$ be a right angle. Let CQ be an altitude to $\triangle ABC$.



Let S, r, S_1, r_1 and S_2, r_2 be the area of the triangle and the radius of the inscribed circle for $\triangle ABC$, $\triangle BQC$ and $\triangle CQA$ respectively.

Now $\triangle BQC \mid\mid\mid \triangle BCA \mid\mid\mid \triangle CQA$ (with corresponding sides in order) by "AAA" by common angles, right angles and angle sum of a triangle.

Since $\triangle BQC \mid\mid\mid \triangle BCA$ then

$$\frac{x}{a} = \frac{z}{b} = \frac{a}{c}$$

so

$$x = a \cdot \frac{a}{c}, \quad z = b \cdot \frac{a}{c}$$

and so

$$S_1 = \frac{1}{2}xz = \frac{1}{2}ab\left(\frac{a}{c}\right)^2 = S\left(\frac{a}{c}\right)^2$$

$$\text{and} \quad a + x + z = a + a\left(\frac{a}{c}\right) + b\left(\frac{a}{c}\right) = (c + a + b)\left(\frac{a}{c}\right).$$

Therefore,

$$r_1 = \frac{2S_1}{a + x + z} = \frac{2S\left(\frac{a}{c}\right)^2}{(c + b + a)\left(\frac{a}{c}\right)} = r\left(\frac{a}{c}\right).$$

Similarly, as $\triangle CQA \mid\mid\mid \triangle BCA$,

$$\frac{y}{b} = \frac{z}{a} = \frac{b}{c}$$

so

$$y = b \cdot \frac{b}{c}, \quad z = a \cdot \frac{b}{c}$$

and so

$$S_2 = \frac{1}{2}yz = \frac{1}{2}ba\left(\frac{b}{c}\right)^2 = S\left(\frac{b}{c}\right)^2$$

$$\text{and} \quad b + z + y = b + a\left(\frac{b}{c}\right) + b\left(\frac{b}{c}\right) = (c + a + b)\left(\frac{b}{c}\right).$$

Consequently,

$$r_2 = \frac{2S_2}{b+z+y} = \frac{2S\left(\frac{b}{c}\right)^2}{(c+a+b)\left(\frac{b}{c}\right)} = r\left(\frac{b}{c}\right).$$

Hence the sum of the areas of the inscribed circles in $\triangle BCQ$ and $\triangle ACQ$ is

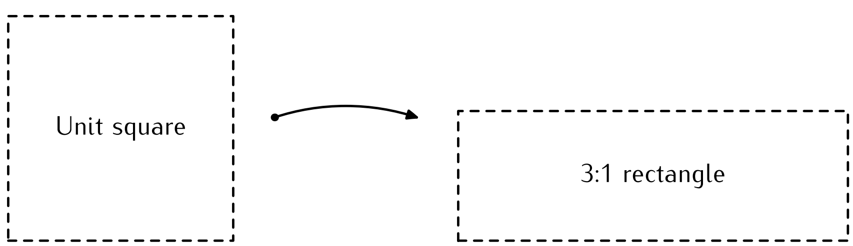
$$\begin{aligned}
\pi r_1^2 + \pi r_2^2 &= \pi r^2\left(\frac{a}{c}\right)^2 + \pi r^2\left(\frac{b}{c}\right)^2 \\
&= \frac{\pi r^2}{c^2}(a^2+b^2) \\
&= \frac{\pi r^2}{c^2}\cdot c^2 \qquad \text{(by Pythagoras' Theorem)} \\
&= \pi r^2 \\
&= \text{Area of the inscribed circle for } \triangle ABC
\end{aligned}$$

Hence after **any number**, say n steps where, at each step, each sub right-angle triangle is subdivided into two sub rightangle triangles by an altitude, the sum of areas of the inscribed circles in the resulting 2^n final rightangle triangles is equal to the area of the original inscribed triangle for the original triangle $\triangle ABC$, i.e.

$$\begin{aligned}
\text{Area} &= \pi\left(\frac{2S}{a+b+c}\right)^2 \\
&= \pi\left(\frac{ab}{a+b+c}\right)^2
\end{aligned}$$

where the sides are a, b, c and c is the length of the hypotenuse. □

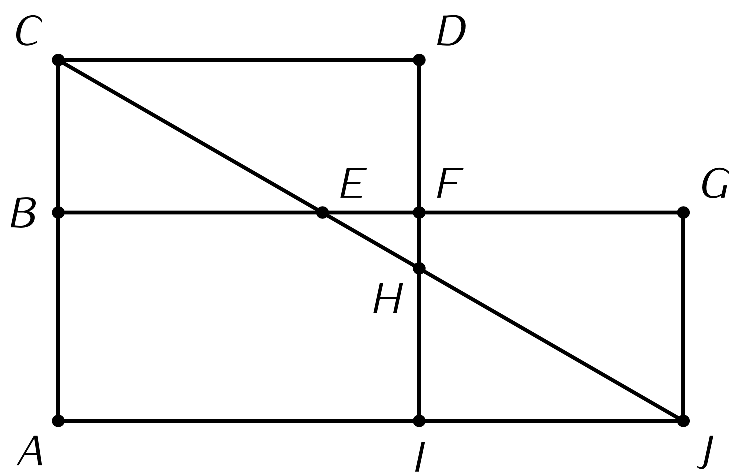
Problem 6
Show how to cut a square of side length 1 by straight lines, so that the resulting pieces can be assembled to form a rectangle in which the ratio of sides is $3:1$.



Proof. [Solution 1] Since we start with a unit square of area 1, and the desired rectangle has sides in ratio 3:1, if the sides are x and $3x$ then

$$1 = 3x^2 \quad \Rightarrow \quad x = \frac{1}{\sqrt{3}}.$$

Go up $\frac{1}{\sqrt{3}}$ on one side of the unit square and draw a line parallel to the other two sides of length $\sqrt{3}$. Complete the $\frac{1}{\sqrt{3}} \times \sqrt{3}$ rectangle $BGJA$ as shown in the diagram.



Now draw the line CJ, the longest line segment from a vertex of the unit square to a vertex of the rectangle.

We claim $\triangle CBE \equiv \triangle HIJ$ and $\triangle CDH \equiv \triangle EGJ$, and hence to perform the transformation, we make two straight line cuts in the unit square along CH and BE, then slide down $\triangle CDH$ on the line CH to the position of $\triangle EGJ$ and move $\triangle CBE$ to the position of $\triangle HIJ$.

We could verify the claim in various ways — here we use coordinate geometry. Let AC be the positive y-axis and AJ the positive x-axis with A the origin. Hence we have coordinates:

$$A(0,0),\ C(0,1),\ D(1,1),\ I(1,0)$$
$$B\left(0,\frac{1}{\sqrt{3}}\right),\ F\left(1,\frac{1}{\sqrt{3}}\right),\ G\left(\sqrt{3},\frac{1}{\sqrt{3}}\right),\ J(\sqrt{3},0)$$

Line CJ has equation

$$y-0=\frac{1-0}{0-\sqrt{3}}(x-\sqrt{3}) \iff y=-\frac{1}{\sqrt{3}}(x-\sqrt{3}).$$

Hence at E, $y=1/\sqrt{3}$ and so

$$\frac{1}{\sqrt{3}}=-\frac{x}{\sqrt{3}}+1 \leftrightarrow x=\sqrt{3}-1 \Rightarrow E\left(\sqrt{3}-1,\frac{1}{\sqrt{3}}\right)$$

and at H, $x=1$ and so

$$y=-\frac{1}{\sqrt{3}}(1-\sqrt{3})=1-\frac{1}{\sqrt{3}} \Rightarrow H\left(1,1-\frac{1}{\sqrt{3}}\right).$$

Hence $\triangle CBE \equiv \triangle HIJ$ as

1. $\angle CBE = \angle HIJ =$ a right angle.
2. $CB = HI = 1-\frac{1}{\sqrt{3}}$.

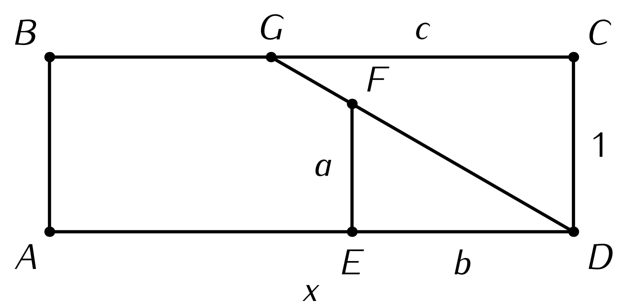
3. $BE = JI = \sqrt{3} - 1$.

And $\triangle CDH \equiv \triangle EGJ$ as

1. $\angle CDH = \angle EGJ =$ a right angle.
2. $CD = EG = 1$.
3. $DH = GJ = \frac{1}{\sqrt{3}}$.

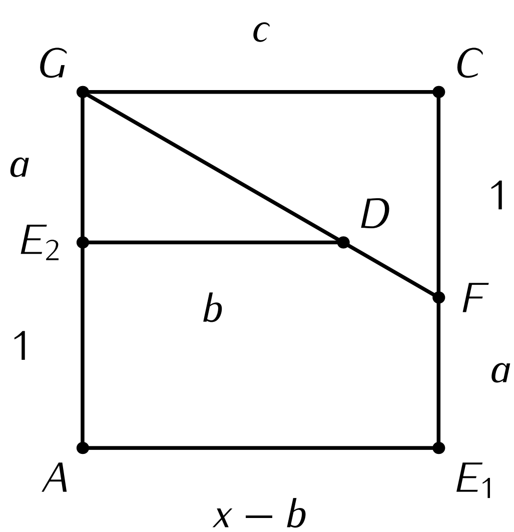
□

Proof. [Solution 2] This solution is essentially the same as a general two-cut construction to convert a rectangle with sides x and 1 into a square where $1 \leq x \leq 4$, as in Joseph S. Madachy, *Madachy's Mathematical Recreations* (Dover Publications, 1979) on page 12 in Chapter 1 *Geometric Dissections*.



$$AD = x,\ CD = 1,\ ED = b,\ EF = a,\ GC = c$$

Make two straight line cuts, GD and EF, then slide up $\triangle GCD$ up and left along line GD, and move $\triangle FED$ up and left into position of $\triangle GE_2D$ in the square.



For consistency of slope of line GD in the rectangle (or $\triangle GCD \parallel\!| \triangle DEF$), we must have

$$\frac{a}{b} = \frac{1}{c}$$

and to obtain a square we must have

$$c = x - b = 1 + a.$$

Therefore,

$$\begin{aligned} b = a + a^2 \ \text{ and } \ x &= 1 + a + b \\ &= 1 + a + a + a^2 \\ &= 1 + 2a + a^2 \\ &= (1 + a)^2 \end{aligned}$$

so

$$a = \sqrt{x} - 1$$

and so we need

$$\begin{aligned} x \geq 1 \ \text{ and } \ b = a + a^2 &= (\sqrt{x} - 1) + x - 2\sqrt{x} + 1 \\ &= x - \sqrt{x}, \\ c = 1 + a &= \sqrt{x}. \end{aligned}$$

This works provided also that

$$c \leq x \ \text{ and } \ a \leq 1.$$

Now,

$$c = \sqrt{x} \leq x \quad \text{iff} \quad x \geq 1 \ \text{ and}$$

$$a = \sqrt{x} - 1 \leq 1 \quad \text{iff} \quad \sqrt{x} \leq 2 \quad \text{or} \quad x \leq 4.$$

□

2014 UNSW School Mathematics Competition
Senior Division - Problems and Solutions

Solutions by Denis Potapov, UNSW, Australia.

Problem 1
The integer part of the real number x, written $[x]$, is the unique integer m, such that

$$m \leq x < m + 1.$$

For example,

$$\left[3 + \frac{1}{2}\right] = 3 \text{ and } \left[-3 - \frac{1}{2}\right] = -4.$$

Let k and n be positive integers. Evaluate the expression

$$\left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] + \cdots + \left[\frac{n+k-1}{k}\right].$$

Proof. [Solution] We set

$$A_n = \left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] + \cdots + \left[\frac{n+k-1}{k}\right].$$

Now, we note that

$$A_0 = \left[0\right] + \left[\frac{1}{k}\right] + \cdots + \left[\frac{k-1}{k}\right] = 0 + 0 + \cdots + 0 = 0.$$

Also, we note that

$$A_{n+1} = A_n - \left[\frac{n}{k}\right] + \left[\frac{n}{k} + 1\right] = A_n + 1.$$

Consequently,

$$A_n = n.$$

□

Problem 2
Players A and B play the following game:

1. the game starts with 1000 counters;
2. at every move, a player subtracts n counters, where n is some power of 2, including $2^0 = 1$;
3. the player cannot subtract more counters than are present at any given stage;
4. the player who first reaches 0 is the winner.

Find the optimal strategy and the winner, if player A starts the game.

Proof. [Solution] Player A always wins by adhering to the strategy in which the number of counters available before every move of player B is to be a multiple of 3:

1. at the start of the game, the number of counters, P, is
$$P \equiv 1 \mod 3;$$
2. if the number of counters before each move of player B is a multiple of 3, then the number of counters before the following move of player A is either
$$P \equiv 1 \mod 3 \text{ or } P \equiv 2 \mod 3;$$
3. by subtracting either
$$2^0 = 1 \text{ or } 2^1 = 2,$$
player A makes sure that the number of counters available to player B on his next move is again a multiple of 3.

□

Problem 3

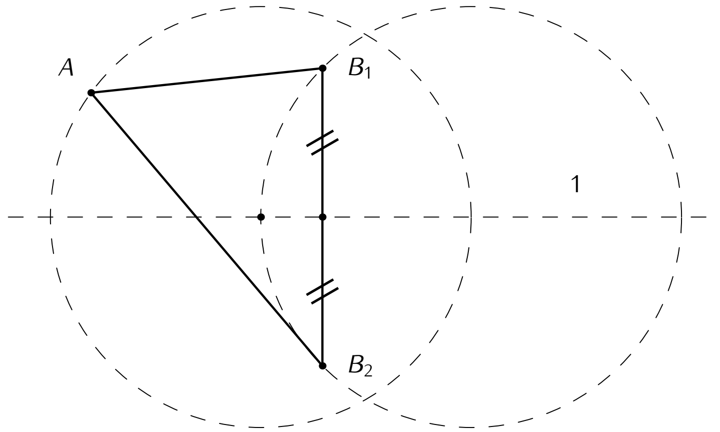
Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n = 1$. Prove that
$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} \geq n^2.$$

Proof. [Solution] By the inequality between arithmetic and geometric mean,
$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \text{ and}$$
$$\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq \left(\frac{1}{a_1 a_2 \cdots a_n}\right)^{\frac{1}{n}}.$$
Multiplying together,
$$(a_1 + a_2 + \cdots + a_n) \times \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$
That is,
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq n^2,$$
given
$$a_1 + a_2 + \cdots + a_n = 1.$$

□

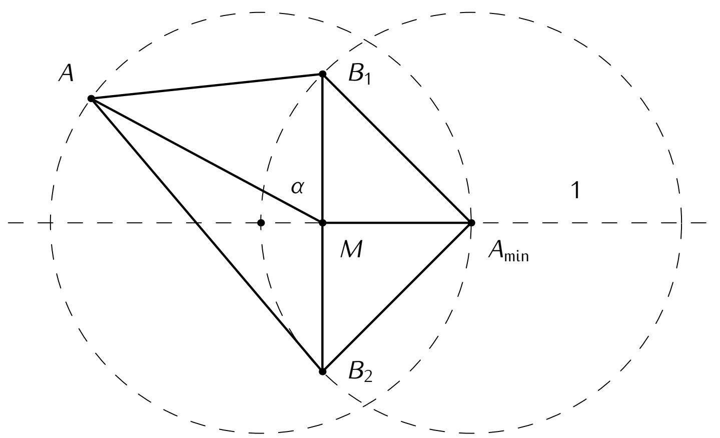
Problem 4

Given two circles of radius 1 with their centres one unit apart, a point A is chosen on the first circle. Two other points B_1 and B_2 are chosen on the second circle, so that they are symmetric with respect to the line connecting the centres of the circles. Prove that
$$(AB_1)^2 + (AB_2)^2 \geq 2.$$



Proof. [Solution] By cosine theorem,

$$AB_1^2 = MA^2 + B_1M^2 - 2MA \times B_1M \times \cos\alpha \text{ and}$$
$$AB_2^2 = MA^2 + B_2M^2 + 2MA \times B_2M \times \cos\alpha.$$



Adding together

$$AB_1^2 + AB_2^2 = 2 \times \left(MA^2 + B_1M^2\right).$$

On the other hand, the distance

$$MA$$

is minimised if

$$A = A_{\text{min}},$$

so by Pythagoras' Theorem

$$MA_{\text{min}}^2 + MB_1^2 = 1.$$

□

Problem 5
Let n be a positive integer.

a) Explain why the set $S = \{1, 2, \ldots, n\}$ can be partitioned into two non-empty disjoint subsets in exactly $2^{n-1} - 1$ ways.

b) Find the number of ways the set $\{1, 2, \ldots, 7\}$ can be partitioned into three non-empty disjoint subsets.

Proof. [Solution] a) There are 2^n subsets of the set

$$S = \{1, 2, \ldots, n\}.$$

If A is one of these, then

$$A,\ S \setminus A$$

gives a partition. Since order is unimportant, there are 2^{n-1} such partitions. Since, the partition

$$\emptyset,\ S$$

is also included in the above computation, the total number of non-empty partitions is $2^{n-1} - 1$.

b) Let a_n be the number of partitions of set

$$S_n = \{1, 2, \ldots, n\}$$

into three non-empty subsets. We see a recurrence relation for the sequence

$$\{a_n\}_{n=1}^{\infty}$$

as follows: we can partition the set

$$S_{n-1} = \{1, 2, \ldots, n-1\}$$

in a_{n-1} ways and there are 3 subsets in which to place the number n. Also, we could partition S_n as

$$A, B, \{n\},$$

where

$$A \cup B = S_{n-1} \text{ and } A \neq \emptyset,\ B \neq \emptyset.$$

By part a), we have exactly

$$2^{n-1} - 1$$

possibilities for the latter. So, we arrive at

$$a_1 = a_2 = 0,\ \ a_3 = 1,\ \ a_n = 3a_{n-1} + 2^{n-2} - 1,\ \ n \geq 4.$$

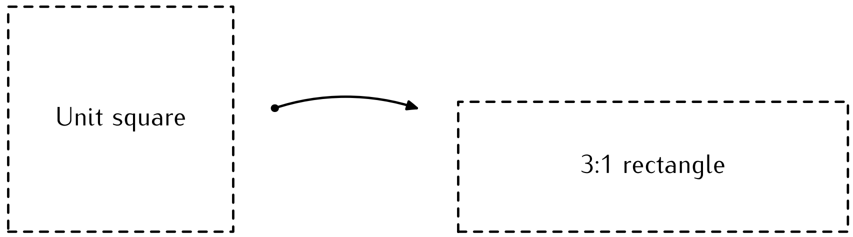
Hence,

$$\begin{aligned} a_4 &= 3a_3 + 4 - 1 = 6, \\ a_5 &= 3a_4 + 8 - 1 = 25, \\ a_6 &= 3a_5 + 16 - 1 = 90, \\ a_7 &= 3a_6 + 31 - 1 = 301. \end{aligned}$$

□

Problem 6

Show how to cut a square of side length 1 by straight lines, so that the resulting pieces can be assembled to form a rectangle in which the ratio of sides is $3 : 1$.



Proof. [Solution] See solution in Junior section. □