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2015 UNSW School Mathematics Competition Junior Division – Problems and Solutions

Solutions by **Denis Potapov**[1]

Problem 1
Every point on a line is painted using two different colours: black and white. Prove that there are always points A_1, A_2 and A_3 of the same colour such that

$$A_1A_2 = A_2A_3\,.$$

Solution. Choose any two points of the same colour, say black, X and Y. Let now A be the centre of XY; B be such that X is the centre of BY and C be such that Y is the centre of CX. Hence, we have the following possibilities:

1. If A is black, then A, X and Y make the desired triple of points.
2. Otherwise, if B is black, then B, X and Y make the desired triple.
3. Otherwise, if C is black, then C, X and Y make the desired triple.
4. Otherwise, A, B and C make the desired triple.

Problem 2
Each of the 64 squares of a chess board has its centre marked. Is it possible to split the board in parts by 13 straight lines such that every part has only one of the points marked?
Note: If a marked centre ends up on a splitting line, then it is assumed that it belongs to both parts of the board on each side of the splitting line.

Solution. Consider 28 squares on all four sides of the board and join adjacent centres of those squared by line segments. There are 28 segments all together. Eeach of the 13 splitting lines intersects at most two such line segments, so there will be at least one segment which is not crossed by a line. Hence, the end points of such segments are in the same part of the board.

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Problem 3
Solve the equation

$$\sqrt{4x+5} - \sqrt{3x+16} = \sqrt{7x-13} - \sqrt{6x-2}\,.$$

Solution. Write the equation in the form

$$\sqrt{4x+5} + \sqrt{6x-2} = \sqrt{7x-13} + \sqrt{3x+16}\,.$$

In such form, since

$$(4x+5) + (6x-2) = (7x-13) + (3x+16),$$

after squaring of both sides, we arrive at

$$\sqrt{4x+5} \times \sqrt{6x-2} = \sqrt{7x-13} \times \sqrt{3x+16}\,.$$

Squaring again gives

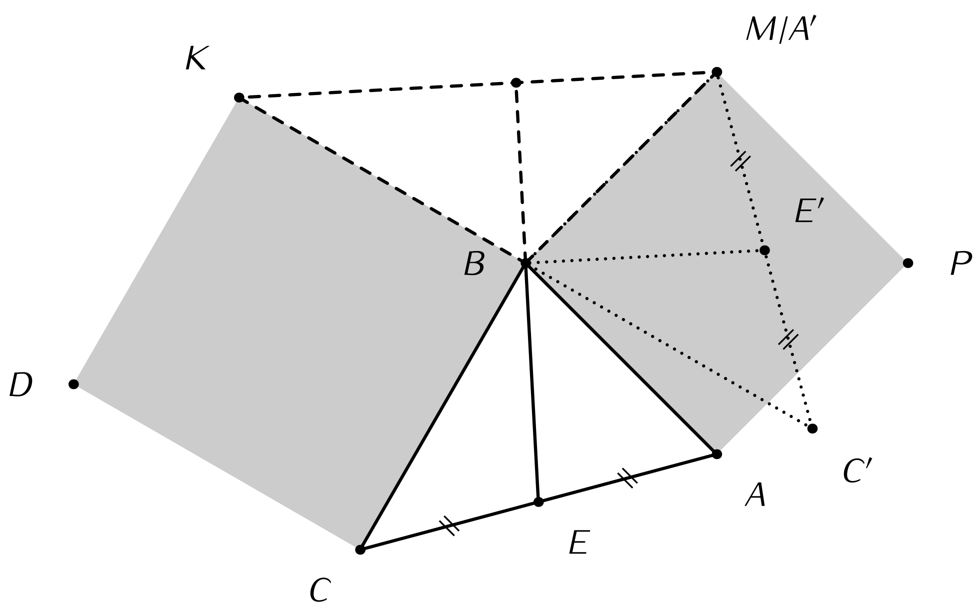
$$(4x+5) \times (6x-2) = (7x-13) \times (3x+16) \quad \Leftrightarrow \quad x^2 - 17\,x + 66\,.$$

The latter solves to

$$x = 6 \text{ and } x = 11\,.$$

Problem 4
A triangle $\triangle ABC$ has squares $ABMP$ and $BCDK$ built on its outer sides. Prove that the median BE of the triangle $\triangle ABC$ is also an altitude of the triangle $\triangle BMK$.

Solution. Rotate the triangle $\triangle ABC$ by 90^{o} around vertex B as shown on the picture below. After such transformation, the median BE becomes the mid-segment of the triangle $\triangle KMC'$. That is, on one hand, BE' is parallel to KM, and on the other hand, it is perpendicular to the median BE.



Problem 5
Find a five-digit number which equals 45 times the product of its digits.

Solution. Let

$$N = \overline{abcde}$$

be the number. We are given that

$$\overline{abcde} = 45\,a\,b\,c\,d\,e\,.$$

Note first that every digit a, b, c, d and e is odd. Indeed, otherwise, N is a multiple of 10 and hence $e = 0$ and $N = 0$.

So, every digit a, b, c, d and e is odd. Since N is a multiple of 5, it follows that $e = 5$. Furthermore, since N is a multiple of 25, then $d = 7$ (the case $d = 2$ is not allowed, since every digit is odd). On the other hand, since N is a multiple of 9, the value

$$a + b + c + 12$$

is also a multiple of 9. Hence, the options are

$$\begin{aligned} a+b+c+12 = 18 \quad &\Leftrightarrow \quad a+b+c = 6 \\ a+b+c+12 = 27 \quad &\Leftrightarrow \quad a+b+c = 15 \\ a+b+c+12 = 36 \quad &\Leftrightarrow \quad a+b+c = 24 \end{aligned}$$

However $a + b + c$ is odd, hence

$$a + b + c = 15\,.$$

Finally,

$$45 \times 35 \times abc \leq 100\,000$$

so

$$abc \leq 63\,.$$

Thus, the choices for a, b and c are narrowed down to:

a, b, c in some order	$\Rightarrow$	abc	$45 \times 35 \times abc$
$9,\ 5,\ 1$	$\Rightarrow$	45	$70\,875$
$9,\ 3,\ 3$	$\Rightarrow$	81	$abc > 63$
$7,\ 7,\ 1$	$\Rightarrow$	49	$77\,175$
$7,\ 5,\ 3$	$\Rightarrow$	105	$abc > 63$

So a solution is

$$N = 77\,175\,.$$

Problem 6

For two rectangles A and B on a plane, we write $A \subseteq B$, if the rectangle A can be placed inside the rectangle B. For instance, if A is a rectangle with sides 2×1, B is a rectangle with sides 2×2, and C is a rectangle with sides 1×3, then we can write $A \subseteq B$ and $A \subseteq C$. However, we can write neither $B \subseteq C$ nor $C \subseteq B$.

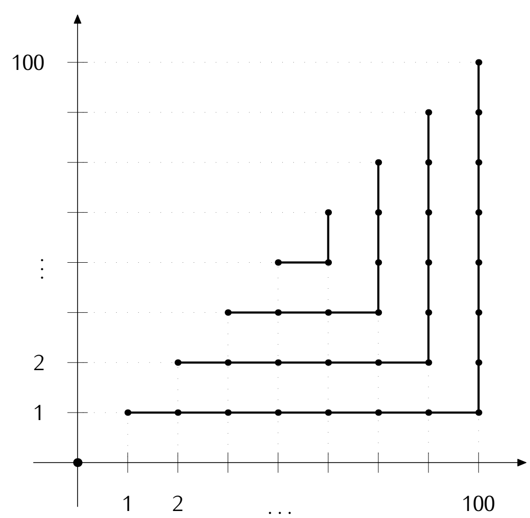
(a) Prove that among 101 rectangles on a plane with integer sides not exceeding 100, there are three rectangles A, B and C such that

$$A \subseteq B \subseteq C\,.$$

(b) Prove that among 2015 rectangles with integer sides not exceeding 100, there are 41 rectangles $A_1, A_2, \ldots, A_{41}$ such that

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_{41}\,.$$

Solution. For every rectangle with sides a and b ($a \leq b$) place a point with coordinates (b, a) on the plane. All possible rectangles with integer sides not exceeding 100 are shown on the following picture. We group these rectangles into 50 disjoint chains as shown on the picture.



Within each chain, the rectangles are completely ordered by the relation $A \subseteq B$. Hence, since

$$2 \times 50 < 100\,,$$

for any 101 rectangles, there are three which end up on the same chain, and therefore, these three form a completely ordered subset.

Similarly, in case of 2015 rectangles, since

$$40 \times 50 < 2015$$

among 2015 rectangles, there are 41 placed on the same chain and therefore completely ordered.

2015 UNSW School Mathematics Competition
Senior Division - Problems and Solutions

Solutions by **Denis Potapov**[2]

Problem 1
A spherical planet has 37 satellites. Prove that there is always a point on the surface of the planet such that at most 17 satellites are seen from this point.

Solution. Fix any two satellites, say S_1 and S_2, and construct the plane through these satellites and the centre of the planet. Let A and B be the end points of the diameter of the planet perpendicular to this plane. The group of satellites visible from A does not intersect with the group of satellites visible from B. Moreover, the satellites S_1 and S_2 are also not visible from both point A and point B. Thus, at most

$$17 = [(37 - 2)/2]$$

are visible from either point A or point B.

Problem 2
The sequence of numbers $\{a_k\}_{k=1}^{\infty}$ is such that

$$a_1 = 1$$

$$\text{and} \qquad a_{k+1} \geq a_k + \frac{1}{a_k}, \quad k = 2, 3, \ldots$$

Prove that $a_{100} > 14$.

Solution. Since $a_{k+1} - a_k \geq \dfrac{1}{a_k}$, it follows that

$$a_{k+1}^2 - a_k^2 = (a_{k+1} - a_k) \times (a_{k+1} + a_k) \geq \frac{1}{a_k}(a_{k+1} + a_k) = \frac{a_{k+1}}{a_k} + 1\,.$$

Since the sequence $\{a_k\}_{k=1}^{\infty}$ consists of positive numbers, we see that

$$a_{k+1} \geq a_k + \frac{1}{a_k} > a_k\,,$$

so $\dfrac{a_{k+1}}{a_k} > 1$. Thus,

$$a_{k+1}^2 - a_k^2 \geq \frac{a_{k+1}}{a_k} + 1 > 2$$

and

$$a_{100}^2 - a_1^2 = (a_{100}^2 - a_{99}^2) + (a_{99}^2 - a_{98}^2) + \cdots + (a_2^2 - a_1^2) > 99 \times 2 = 198$$

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so

$$a_{100} > \sqrt{196} > 14\,.$$

Problem 3

You are given a square table filled with positive integers. On every move, you are allowed to take 1 from every element of a row; or to multiply every element of a column by 2. Prove that there is a strategy which can reduce every element in the table to zero.

Solution. We call a row a "zero row" if it consists of all zero entries; we call a row a "non-zero row" if it consists of all non-zero entries; we call a row a "partially zero row" if it has some (but not all) zero entries.

(a) Assume that the table has some zero rows. In this case, applying the allowed operations on another row (a non-zero row or a partially zero row) does not affect the existing zero rows.

(b) Originally, every row is non-zero. Let us show that every non-zero row can be converted to a zero row such that a partially non-zero row never appears. This, together with part (a) finishes the solution to this question.

Indeed, fix a non-zero row, say R. Firstly, every column which has entry exactly 1 in the row R is multiplied by 2. Secondly, every entry of the row R is reduced by 1. As a result, every entry of the row R which had value 1, kept the value 1. Every other entry of the row R is reduced by 1.

By sufficiently repeating the above strategy, every entry of the row R can be reduced to value 1 without making the row R a partially zero row during this reduction.

Finally, when the row R consists of all 1's, we reduce to zero each entry of the row R by one move of the first type.

Problem 4

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$$a + b + c + 12$$

is also a multiple of 9. Hence, the options are

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$$a+b+c+12=27 \quad \Leftrightarrow \quad a+b+c=15$$
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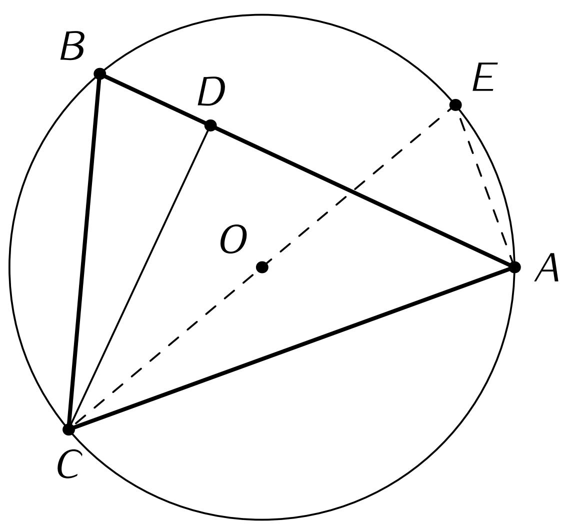
Problem 5
The radius of the circumscribed circle of a triangle is $\frac{65}{6}$. Find the third side of the triangle if the other two are 20 and 13 and every angle is acute.

Solution. Let

$$AC = 20 \text{ and } BC = 13 \text{ and } OC = \frac{65}{6}\,.$$



Let CD be the height of $\triangle ABC$ and let CE be the diameter of the circle. Connect points A and E. The triangle $\triangle CAE$ is right-angled since the angle $\angle CAE$ is rested on a diameter. Hence, the triangles

$$\triangle ACE \text{ and } \triangle CDB$$

are similar. The similarity implies that

$$\frac{CD}{CB} = \frac{AC}{CE}\,.$$

Solving for CD gives

$$CD = \frac{CB \times AC}{CE} = \frac{13 \times 20}{2 \times \frac{65}{6}} = 12\,.$$

By Pythagoras' Theorem applied to $\triangle CDB$,

$$DB = \sqrt{13^2 - 12^2} = 5$$

and, by By Pythagoras' Theorem on $\triangle ADC$,

$$AD = \sqrt{20^2 - 12^2} = 16\,.$$

Thus,

$$AB = 16 + 5 = 21\,.$$

Problem 6

Let $F(x)$ be a polynomial with integer coefficients and let

$$a_1, a_2, \ldots, a_m$$

be integers such that for any $n \in \mathbb{N}$, there is an a_i such that $F(n)$ is a multiple of a_i. Prove that there is one a_i such that $F(n)$ is a multiple of a_i for any $n \in \mathbb{N}$.
Note: The following theorem is known as **The Chinese Remainder Theorem** and it can be used in the solution of this problem without proof.

Theorem

Let

$$q_1, q_2, \ldots, q_r \in \mathbb{Z}$$

be positive pairwise coprime integers. For any integers

$$x_1, x_2, \ldots, x_r \in \mathbb{Z}$$

there is an integer $x \in \mathbb{Z}$ such that

$$x \equiv x_i \pmod{q_i}\,, \quad i = 1, 2, \ldots, r\,.$$

Solution. Let us assume the contrary: there is an integer $x_i \in \mathbb{Z}$ such that

$$F(x_i) \not\equiv 0 \pmod{a_i} \;\; \forall i = 1, 2, \dots m\,.$$

Factorising a_i into prime factors, select the factor

$$q_i := p_i^{s_i}\,, \;\; p_i \text{ prime}$$

such that

$$F(x_i) \not\equiv 0 \pmod{q_i}\,, \;\; i = 1, 2, \dots, m\,.$$

In the case of identical primes appearing among the powers q_i's, select the factor with the highest exponent s_i. Hence, we constructed pairwise coprime integers

$$q_1, q_2, \dots, q_r, \;\; r \leq m$$

such that, for any $b \in \mathbb{Z}$,

$$b \not\equiv 0 \pmod{q_i} \; \forall i = 1, 2, \dots, r \qquad \Longrightarrow \qquad b \not\equiv 0 \pmod{a_i} \; \forall i = 1, 2, \dots, m\,.$$

Finally, note that, since F is a polynomial with integer coefficients,

$$F(x_i) \not\equiv 0 \pmod{q_i} \qquad \Longrightarrow \qquad F(x_i + kq_i) \not\equiv 0 \pmod{q_i} \quad \forall k \in \mathbb{Z}\,.$$

Consequently, by the **Chinese Remainder Theorem**, there is $x \in \mathbb{Z}$ such that

$$x = x_i + k_i q_i$$

for some $k_i \in \mathbb{Z}$ and so

$$F(x) \not\equiv 0 \pmod{q_i} \; \forall i = 1, 2, \dots, r \qquad \Longrightarrow \qquad F(x) \not\equiv 0 \pmod{a_i} \; \forall i = 1, 2, \dots, m\,.$$