MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 9, July 1, 2019

1. Find the sum of all \(n\)-digit long numbers formed by 1, 2, 3, \ldots, \(n\). For example, if \(n = 3\) then the sum of all 3-digit long numbers is \(123 + 132 + 213 + 231 + 312 + 321 = 1332\).

2. Evaluate \(\sqrt{2} \times \sqrt[4]{4} \times \sqrt[8]{8} \times \sqrt[16]{16} \times \sqrt[32]{32} \ldots\).

3. Several positive integers are written on a blackboard. The sum of any two of them is some power of two (for example, 2, 4, 8, \ldots). What is the maximal possible number of different integers on the blackboard?

4. For a triangle \(ABC\), \(M\) is the midpoint of the side \(AB\) and \(L\) is some point along the side \(BC\). Let \(O\) be the point of intersection between the lines \(LA\) and \(MC\), and let \(K\) be the point of intersection between \(LA\) and the line passing through \(M\), parallel to \(BC\), as shown in the diagram below.

\[\text{NOT TO SCALE}\]

(a) Show that the triangles \(\triangle KMO\) and \(\triangle LCO\) are similar.

(b) Suppose the length \(LA\) is twice as long as \(MC\), and \(\angle OLC = 45^\circ\). Prove \(LA\) is perpendicular to \(MC\).

5. Consider the polynomial \(p(x) = x^4 + 37x^3 + 71x^2 + 18x + 3\). If \(a, b, c\) and \(d\) are roots of \(p(x)\), find a polynomial whose roots are \(\frac{abc}{d}, \frac{acd}{b}, \frac{abd}{c}\) and \(\frac{bcd}{a}\).
Senior Questions

1. The speed of a multi-pole DC motor is described by the following simplified relation:

\[ j \frac{d^2 \theta}{dt^2} + c \frac{d\theta}{dt} = I_{motor}, \]  

where \( \theta \) is the angular position of the motor, \( j \) is the rotational inertia of the load, \( c \) is the damping coefficient of the load and \( I_{motor} \) is the current through the motor circuit. Additionally, the motor circuit has an associated resistance (\( R \)) and back EMF (\( k_m \frac{d\theta}{dt} \)) described by the following formula:

\[ V_{in} = R \times I_{motor} + k_m \frac{d\theta}{dt}, \]

where \( V_{in} \) is the applied battery voltage. Given \( j = 5 \), \( c = 1 \), \( V_{in} = 12 \) volts, \( R = 10 \) ohms, \( k_m = 5 \), find:

(a) The limiting angular speed of the motor.
(b) The initial acceleration of the motor from a stationary state.
(c) Solve for \( \theta(t) \).

2. Using straight-edge and compass techniques, describe how to construct a triangle given its circumcircle (\( C \)) and the three points on it at which the altitude (\( A \)), the angle bisector (\( B \)), and the median (\( C \)) drawn from the same vertex intersect the circle.