MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 1, May 13, 2019

1. Let $x = 0.284284284\ldots$. Then

$$\begin{aligned} 1000x &= 284.284284284\ldots \\ &= 284 + x. \end{aligned}$$

Thus $x = \frac{284}{999}$.

2. If $x - y + 2z$ is divisible by 11, then there is an integer k such that $x - y + 2z = 11k$. So we can write

$$\begin{aligned} -12x + y - 13z &= -x + y - 2z - 11x - 11z \\ &= -11k - 11(x + z) \\ &= -11(k + x + z). \end{aligned}$$

The right hand side of the above equation is divisible by 11, since $k + x + z$ is an integer. Thus $-12x + y - 13z$ is divisible by 11.

3. The speed of Anna and Boris, and the initial distance between them are constant regardless of when they started moving toward each other. Therefore, we would like to express everything we need to solve this problem in terms of those constants. Let V_A and V_B be the speed of Anna and Boris, respectively, and let x be the initial distance between them. If they move towards each other simultaneously, then the time it takes for Anna and Boris to meet is

$$\frac{x}{V_A + V_B}.$$

This means the distance covered by Anna is $\frac{xV_A}{V_A+V_B}$, and the distance covered by Boris is $\frac{xV_B}{V_A+V_B}$.

Suppose they started moving at a different time, so that the distance covered by Anna is 2km more and the distance covered by Boris is 2km less. Then the time Anne spent moving is $\frac{1}{V_A}\left(\frac{xV_A}{V_A+V_B} + 2\right)$. Similarly, Boris spent $\frac{1}{V_B}\left(\frac{xV_A}{V_A+V_B} - 2\right)$ moving. But we already know the difference in timing is $\frac{1}{2}$ hour, therefore

$$\frac{1}{V_A}\left(\frac{xV_A}{V_A + V_B} + 2\right) - \frac{1}{V_B}\left(\frac{xV_B}{V_A + V_B} - 2\right) = \frac{1}{2},$$

or simplifying to get $\frac{1}{V_A} + \frac{1}{V_B} = \frac{1}{4}$. This expression is symmetric, so if we switch the starting time condition between Anna and Boris, then Anna would cover 2km less and Boris 2km more; $d = 2$.

4. Let x be the four digit number we are trying to find. Then $x^2 - x = x(x-1)$ is a number ending in 0000. That is, $x(x-1)$ is divisible by $10\,000 = 2^4 5^4$. Now x and $x-1$ are coprime, which is to say that they have no prime factors in common. Thus one of $x-1$ or x is divisible by 2^4 and the *other* by 5^4. If x or $x-1$ is divisible by 5^4, then it is an odd multiple of $5^4 = 625$, so x or $x-1$ must be one of

   $$625,\ 1875,\ 3125,\ 4375,\ 5625,\ 6875,\ 8125,\ 9375.$$

   We add or subtract 1 from the above list to find out which of the *other* number is divisible by $2^4 = 16$. The two possibilities are $x - 1 = 624$ and $x = 9376$. We discard the first option as it is not a proper four digit number.

5. We begin by assigning letters to each of the cells in the 3×3 grid.

a	b	c
d	e	f
g	h	i

   Although we do not know the individual value of each letter we do know that each of the digits 1 through 9 is assigned to the letters in some order. Let T be the total by adding each row, column or diagonal, e.g $T = a+b+c$. Then by adding all three rows, we get the number $3T$. Note this is the same as adding every cell.

   $$a+b+c+d+e+f+g+h+i = 1+2+...+9 = 45 = 3T,$$

   so $T = 15$. Suppose instead we add every row, column or diagonal that involves the middle cell, then

   $$\begin{aligned}(a+e+i)+(d+e+f)+(g+e+c)+(b+e+h) &= 4T\\ (a+b+c+d+e+f+g+h+i)+3e45+3e &= 60\\ 45+3e &= 60\\ e &= 5.\end{aligned}$$

   Hence we know that $T = 15$ and the middle cell must be 5. So in order for each line to have the same total of 15 it will be necessary for the cells either side of the central cell to be of the form $5x$ and $5+x$.

$5+x$	$5-x-y$	$5+y$
$5-x+y$	5	$5+x-y$
$5-y$	$5+x+y$	$5-x$

   The cell with greatest value is $5+x+y = 9$, hence $x+y = 4$. Also $x \neq y$, otherwise the cells $5+y$ and $5+x$ would have the same number in them; Finally $x, y > 0$ to avoid the cells $x+5$ and $x-5$ being the same.

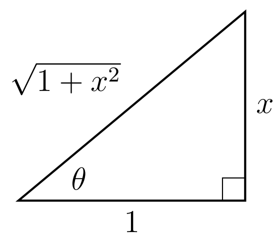
   Because $x \neq y$, we can assume without loss that $x < y$, and since $x + y = 4$, we conclude that $x = 1$ and $y = 3$. Substituting these values into the grid above we obtain the solution given in the problem and hence prove that this solution is unique.

Senior Questions

1. (a) Let $\theta = \tan^{-1}(x)$, then $x = \tan(\theta)$, and

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta}.$$

Now by using the picture below, we can see that $\cos^2\theta = \frac{1}{1+x^2}$, so $dx/d\theta = 1+x^2$.



Thus

$$\frac{d\theta}{dx} = \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2},$$

as required.

(b) Consider the geometric series $1 - x^2 + x^4 - x^6 + \ldots$, which common ratio $r = -x^2$. If $|x^2| < 1$, then we can find the infinite sum of this series and hence

$$1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1+x^2}.$$

Since the LHS of the above equation is $\frac{d}{dx}\tan^{-1}(x)$ by Q1(a), integrating both sides gives

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

2. (a) Rearranging the factors slightly,

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 &= [n(n+3)][(n+1)(n+2)]+1 \\ &= (n^2+3n)(n^2+3n+2)+1 \\ &= (n^2+3n)^2 + 2(n^2+3n)+1 \\ &= (n^2+3n+1)^2 \end{aligned}$$

(b) In this case, let $n = 28$. Then

$$\begin{aligned} \sqrt{(31)(30)(29)(28)+1} &= \sqrt{(28^2+3\times 28+1)^2} \\ &= 28^2 + 3\times 28 + 1 \\ &= 869 \end{aligned}$$