MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 10 Solutions, August 13, 2019

1. Carla wins if either or both players rolls a 5 or 6. Summing the probability of each case to occur gives
$$2 \times \left(\frac{2}{6} \times \frac{4}{6}\right) + \frac{2}{6} \times \frac{2}{6} = \frac{5}{9};$$
Carla is more likely to win.

2. First note that $2^{10} = 1024$, so that the distinct numbers we are adding can have at most a power of 9 on 2. Now adding the ten possible distinct powers of 2, gives $2^0 + 2^1 + \ldots + 2^9 = 2^{10} - 1 = 1023$, and we have to delete a number from the summation on the LHS that is greater than 23 in order to obtain the 3 digit positive integer we need; there is only five distinct powers of 2 that is greater than 23 from the list.

3. By considering the expansion $(a+b)^2 = a^2 + b^2 + 2ab$, and using the fact that $a + b = 1$ and $a^2 + b^2 = 2$, we can deduce that $ab = -1/2$.

   Next by considering the expansion of $(a^{n-1} + b^{n-1})(a+b)$, where n is a positive integer, and using the fact that $a + b = 1$ and $ab = -1/2$, we have
$$\begin{aligned}(a^{n-1} + b^{n-1})(a + b) &= a^n + b^n + a^{n-1}b + ab^{n-1} \\ (a^{n-1} + b^{n-1}) \times 1 &= a^n + b^n - \frac{1}{2}a^{n-2} - \frac{1}{2}b^{n-2} \\ a^n + b^n &= a^{n-1} + b^{n-1} + \frac{1}{2}(a^{n-2} + b^{n-2}).\end{aligned}$$
   The last line of the above equation is a recursive relationship on $a^n + b^n$, so by setting $n = 7$, we can after some work find $a^7 + b^7$, which is $71/8$.

4. Rearranging the equation, we have
$$x - y = \sqrt{x} - \sqrt{y}$$
   We can treat the LHS as a difference of two squares, in which case,
$$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = \sqrt{x} - \sqrt{y}$$

Now $x \neq y$, so we can cancel the common factor of $\sqrt{x} - \sqrt{y}$ to obtain

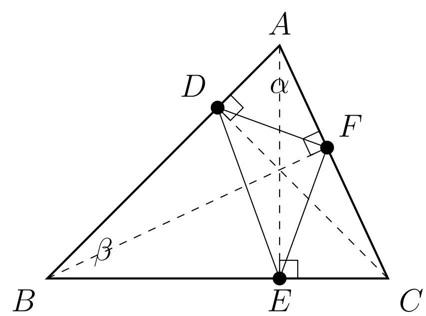
$$\sqrt{x} + \sqrt{y} = 1.$$

Squaring both sides and rearranging, we have

$$\begin{aligned} x + 2\sqrt{xy} + y &= 1 \\ x + y &= 1 - 2\sqrt{xy} \end{aligned}$$

Now the LHS is what we are trying to maximise. If we look at the RHS, we can see that this is maximised if either x or y is zero and the other number must then be one. So the maximum value is one.

5. Let $\angle BAC = \alpha$ and $\angle ABC = \beta$.



(a) Since F and E are the feet of two altitudes, $\angle AFB = \angle BEA = 90^\circ$. Thus $AFEB$ is a cyclic quadrilateral, of which $\angle EFC$ is an exterior angle. Hence $\angle EFC = \angle ABC = \beta$. In a similar fashion, it can be shown that $\angle CEF = \angle BAC = \alpha$. Thus $\triangle EFC$ is similar to $\triangle ABC$ (equiangular).

(b) By a similar argument to that in part (a), we can show that $\angle EFC = \angle DFA = \beta$, which implies that $\angle BFE = \angle BFD = 90^\circ - \beta$. Thus BF is the bisector of $\angle DFE$. Moreover, if we consider a different pair of altitudes, we can show that CD bisects $\angle FDE$ and AE bisects $\angle DEF$.

Senior Questions

1. Since we are dividing by $x^2 - 1$, the remainder is a polynomial of x of at most degree 1; that is the remainder takes the form $ax + b$, for some constants a and b.

   To find a and b, write

$$\begin{aligned} x^{2019} &= Q(x)(x^2 - 1) + ax + b \\ &= Q(x)(x-1)(x+1) + ax + b, \end{aligned}$$

   where $Q(x)$ is a polynomial of x. Then by putting $x = 1$ and $x = -1$ into the last line of the above equation, we have $a + b = 1^{2019} = 1$ and $-a + b = -1^{2019} = -1$. Solving these simultaneously, we arrive at $a = 1$ and $b = 0$.

2. By the sine rule,

$$\begin{aligned} \frac{\sin\theta}{4} &= \frac{\sin 2\theta}{6} \\ 3\sin\theta &= 2\sin 2\theta \\ 3\sin\theta &= 4\sin\theta\cos\theta, \end{aligned}$$

   so

$$\sin\theta = \frac{4}{3}\sin\theta\cos\theta$$

   Since $\sin\theta \neq 0$, we may cancel this to obtain $\cos\theta = \frac{3}{4}$.

   By the cosine rule,

$$\begin{aligned} \cos\theta &= \frac{x^2 + 6^2 - 4^2}{12x} \\ \frac{3}{4} &= \frac{x^2 + 20}{12x} \end{aligned}$$

   Thus $x^2 - 9x + 20 = 0$, which has the solutions $x = 4$ or $x = 5$.

   Now, if $x = 4$, then $\triangle ABC$ is isosceles, and $\angle BAC = \angle BCA = \theta$. So by the angle sum of $\triangle ABC$, $\theta = 45°$, and hence $\angle B$ is a right angle. But $4^2 + 4^2 \neq 6^2$, so this is clearly incorrect.

   That leaves us with $x = 5$ as the only solution.

3. Consider the function $f(x) = 2^x + 3^x + 6^x - x^2 = 0$. By a change of base, we have $f'(x) = \ln 22^x + \ln 33^x + \ln 66^x - 2x$.

   Case 1: If $x < 0$, then $f'(x) > 0$. Therefore $f(x)$ is strictly increasing for $x < 0$, which implies there is only one solution for this case; the unique solution is $x = -1$.

   Case 2: If $x \geq 0$, suppose we have a solution s. Then $s^2 = 2^s + 3^s + 6^s \geq 3$, hence $s \geq \sqrt{3} \geq 1$. Which implies $2^s = (1+1)^s \geq 1 + s \geq s$. Now $6^s > 4^s = 2^{2s} \geq s^2$, so that $2^s + 3^s + 6^s > s^2$, a contradiction.