MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 12 Solutions, August 27, 2019

1. Since $1! + 2! + 3! = 9 = 3^2$, we know that $n = 3$ is a possible solution to the problem. We will show that $n = 3$ is the largest solution. Observe that for any number to be a perfect square, it cannot end in the digit 3. Since $1! + 2! + 3! + 4! = 33$, $n = 4$ is not a solution. Moreover, $n!$ contains factors of both 2 and 5 for $n > 4$, therefore $n!$ ends in the digit 0 for $n > 4$. We can now conclude that the number $1+2!+3!+\ldots+(n-1)!+n!$ ends in the digit 3 for $n > 4$, thus cannot be a perfect square.

2. Since we are adding consecutive integers, we know that $a_2 = a_1 + 1, a_3 = a_1 + 2, \ldots, a_{100} = a_1 + 99$. Therefore, we can write

$$\begin{aligned}\sqrt{a_2 + a_3 + \ldots + a_{99}} - \sqrt{a_1 + a_{100}} &= \sqrt{98a_1 + 1 + 2 + \ldots + 98} - \sqrt{2a_1 + 99} \\ &= \sqrt{98a_1 + 4851} - \sqrt{2a_1 + 99}.\end{aligned}$$

   The second line of the equation above is minimal when $a_1 = 1$.

3. Let abc and efg be three digit numbers. Then we can write the initial 6-digit number x as

$$x = 1000 \times abc + efg.$$

   We also know that

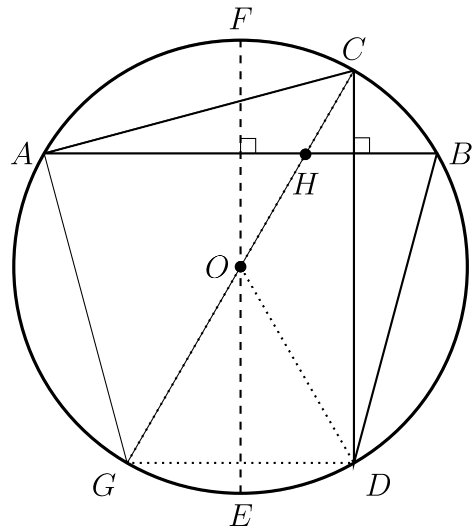
$$6x = 1000 \times efg + abc.$$

   Combining the above two equations gives $5999 \times abc = 994 \times efg$, which can be further simplified to $857 \times abc = 142 \times efg$. Hence the number we are after is 142857.

4. To have all frogs the same colour, we must first reach a situation where there is the same number of frogs of two different colours. So we can think about this problem in terms of the difference between the number of frogs having two different colours, then it is possible to have all frogs the same colour if this number is zero for any combination of two colours. For example, initially this number is 1 if we compare brown with green or green with yellow, and 2 if we compare brown with yellow.

   Now if two frogs with different colours meet, both frogs change colour to the third, so the number of frogs of different colours either changes by 3 or remains the same. Since we started with a difference of 1 or 2, and can only change this number by 3, it is not possible to get the same number of frogs of two different colours.

5. Let EF be the diameter perpendicular to CD, and let G be the reflection of D in the line EF, as shown below.



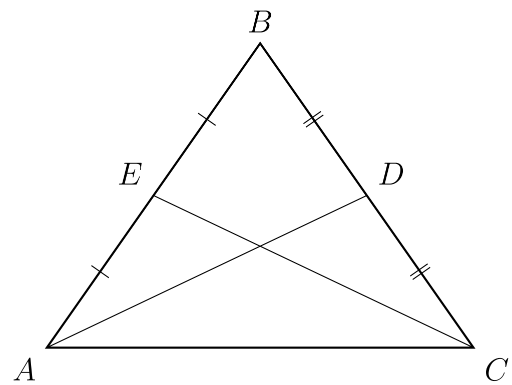
Since G is the reflection of D, $AG = BD$. If we can show that COG is a diameter of the circle, then $\triangle ACG$ is right angled (by Thales' theorem) and the desired result follows by Pythagoras' theorem. In order to show that COG is a diameter of the circle, we must show that $\angle COG$ is straight.

Let H be the point of intersection of OC ad AB. Let $\angle OCD = \alpha$ and $\angle CHB = \beta$. Since $CD \perp AB$, α and β are complementary. Furthermore, $\angle CHB$ and $\angle OHA$ are vertically opposite, so $\angle OHA = \beta$a nd thus $\angle HOF = \alpha$.

Now $\triangle OCD$ is isosceles, thus $\angle ODC = \angle OCD = \alpha$, and since FE is parallel to CD, $\angle EOD = \angle ODC = \alpha$. As G is the reflection of D in the line EF, $\triangle OEG \equiv \triangle OED$, and thus $\angle GOE = \angle DOE = \alpha$. Thus $\angle GOE$ and $\angle HOF$ are equal and thus vertically opposite, and so $\angle GOC$ is straight, as required.

Senior Questions

1. From Problem Sheet 11, Senior Question 1(b), recall that the length of the median can be expressed in terms of the side lengths, $a = BC$, $b = AC$ and $c = AB$.



Thus $EC = \frac{1}{2}\sqrt{2(a^2+b^2)-c^2}$ and $AD = \frac{1}{2}\sqrt{2(c^2+b^2)-a^2}$. So if $EC = AD$, then

$$\begin{aligned} 2(a^2+b^2)-c^2 &= 2(c^2+b^2)-a^2 \\ 3a^2 &= 3c^2 \end{aligned}$$

The only positive solution to this equation is $a = c$, and thus $\triangle ABC$ is isosceles.

2. Using the method of partial fractions, we can write

$$\frac{2n-1}{n(n+1)(n+2)} = \frac{-1}{2n} + \frac{3}{n+1} + \frac{-5}{2(n+2)}.$$

Thus

$$\begin{aligned} \sum_{n=1}^{25} \frac{2n-1}{n(n+1)(n+2)} &= \sum_{n=1}^{25} \frac{-1}{2n} + \frac{3}{n+1} + \frac{-5}{2(n+2)} \\ &= \sum_{n=1}^{25} \frac{-1}{2n} + \sum_{n=1}^{25} \frac{3}{n+1} + \sum_{n=1}^{25} \frac{-5}{2(n+2)} \\ &= -\frac{1}{2}\sum_{n=1}^{25} \frac{1}{n} + 3\sum_{n=2}^{26} \frac{1}{n} - \frac{5}{2}\sum_{n=3}^{27} \frac{1}{n} \\ &= \left(-\frac{1}{2} + 3 - \frac{3}{2}\right)\sum_{n=3}^{25} \frac{1}{n} - \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2}\right) + 3\left(\frac{1}{2} + \frac{1}{26}\right) - \frac{5}{2}\left(\frac{1}{26} + \frac{1}{27}\right) \\ &= \frac{475}{702}. \end{aligned}$$