MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13 Solutions, September 3, 2019

1. We can write

$$\begin{aligned} y &= 1000 \times x + x \\ &= 1001x \end{aligned}$$

Also, $y = kx^2$ for some integer k. Thus

$$1001x = kx^2$$

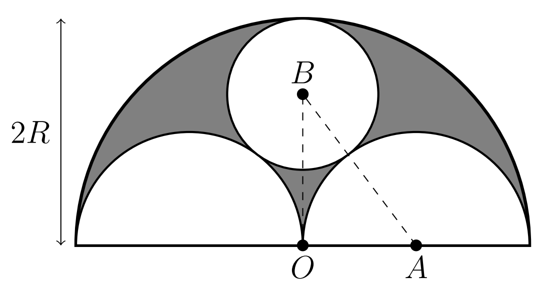
Since $x \neq 0$, we may divide both sides by x to obtain

$$\begin{aligned} kx &= 1001 \\ &= 7 \times 143 \end{aligned}$$

Since x is a three-digit number, $x = 143$ and $y = 143143$.

2. If we calculate the fifth powers in mod 10, we see that $1^5 \equiv 1 \pmod{10}$, $2^5 \equiv 2 \pmod{10}$, $3^5 \equiv 3 \pmod{10}$, and so on. That is, the last digit of x^5 will be the same as the last digit of x. Therefore, the last digit of $1^5 + 2^5 + \ldots + 2019^5$ is equal to the last digit of $1 + 2 + \ldots 2019$, which is 0.

3. Let O be the centre of the big semi-circle; A be the centre of one of the smaller semi-circles, and B be the centre of the inscribed circle.



Clearly, $OA = R$. Now AB is a straight line through the point of tangency of the circle and the semi-circle, so $AB = R + r$. Since the two smaller semi-circles are congruent, and the circle is tangent to both of them, B lies on the vertical axis of symmetry of the larger semi-circle. Thus $OB = 2R - r$.

Thus, by Pythagoras' theorem,

$$\begin{aligned}(R+r)^2 &= (2R-r)^2 + R^2 \\ R^2 + 2rR + r^2 &= 4R^2 - 4rR + R^2 + R^2 \\ 6rR &= 4R^2 \\ 3r &= 2R\end{aligned}$$

Hence $R : r = 3 : 2$, as required.

4. Suppose we can place the numbers on a circle so that the condition holds. Let us call the integers from 26 to 75 *normal*, and all the others *extreme*. Two extreme integers cannot be consecutive (their difference is either less than 25 or greater than 50). Note that the numbers of the extreme and normal integers are the same and therefore they must alternate. However the normal number 26 can be adjacent to only one extreme integer 76, which is a contradiction.

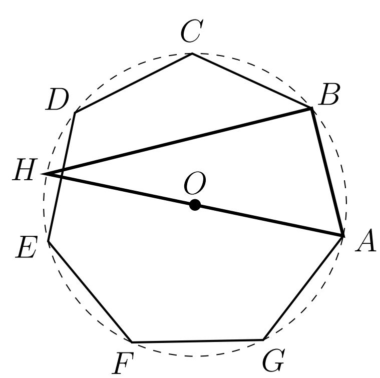
5. Let $a = 10^b$, then we can rewrite the inequality $10 < a^x < 100$ as $1 < bx < 2$. Similarly, if $100 < a^x < 1000$, then $2 < bx < 3$. Suppose n is the smallest integral solution to the inequality, then since there are exactly 5 solutions, the largest solution must be $n+4$. From this we can deduce $b(n-1) < 1 < bn$ and $b(n+4) < 2 < b(n+5)$. Summing up the first inequality with itself and with the second one we obtain $b(2n-2) < 2 < b(2n)$ and $b(2n+3) < 3 < b(2n+5)$. Therefore, the inequality $2 < bx < 3$ has from 4 to 6 integer solutions; $2n, 2n+1 \ldots, 2n+4$ are always solutions, while $2n-1$ and $2n+4$ may or may not be.

   So if we want to get a only four solutions, then we need to consider a number b such that $b(2n-2) < 2$ and $3 < b(2n+5)$ for some integer n. An easy way to do this is set $n = 5$, then $\frac{1}{5} < b < \frac{1}{4}$. The solutions for the first inequality is $5, 6, 7, 8$ and the second $10, 11, 12, 13$.

   We can get 5 or 6 solutions by picking the appropriate b.

Senior Questions

1. Without loss of generality, we can inscribe the regular heptagon in a circle centered at O with diameter 1 unit. Let H be the point on the circle opposite A, so that AOH is a diameter of the circle. Then AH is one unit.



Now, consider $\triangle ABH$. As AH is a diameter of the circle, $\angle HBA = \frac{\pi}{2}$. Furthermore, since $ACDEFG$ is a regular heptagon, $\angle AOB = \frac{2\pi}{7}$, which implies that $\angle BHA = \frac{\pi}{7}$. Thus $AB = \sin\frac{\pi}{7}$. By a similar argument, we can show that $AC = \sin\frac{2\pi}{7}$ and $AD = \sin\frac{3\pi}{7}$. Thus if we can show that

$$\csc\frac{\pi}{7} = \csc\frac{2\pi}{7} + \csc\frac{3\pi}{7},$$

then the desired result is proved.

So

$$\begin{aligned}\csc\frac{2\pi}{7} + \csc\frac{3\pi}{7} &= \frac{1}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{3\pi}{7}}\\ &= \frac{\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}\\ &= \frac{\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\sin\frac{3\pi}{7}}\end{aligned}$$

Now we can make use of the fact that $\sin\frac{3\pi}{7} = \sin\left(\pi - \frac{3\pi}{7}\right) = \sin\frac{4\pi}{7}$ and the incredibly useful products-to-sums trig identity

$$2\sin A\cos B = \sin(A-B) + \sin(A+B),$$

to show that

$$\begin{aligned}\frac{\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\sin\frac{3\pi}{7}} &= \frac{\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7}}{\sin\frac{\pi}{7}(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7})}\\ &= \frac{1}{\sin\frac{\pi}{7}},\end{aligned}$$

as required.

2. Notice that we have the greatest control over the number p_1, so we want to find out what p_1 is allow to be. Suppose $p_1 > 3$, then $p_1, \dots p_{17}$ can not contain factors of 3. Therefore, $p_i = 1 \pmod 3$ or $p_i = 2 \pmod 3$ for $i = 1, 2, \dots, 17$; that is $p_1, \dots, p_{17}$ must have remainder 1 or 2 when divided by 3. From this, we have $p_i^2 = 1 \pmod 3$ for each $i = 1, \dots 17$, and so $p_1^2 + p_2^2 + \dots + p_{17}^2 = 2 \pmod 3$. On the other hand, the square of an integer must have remainder 0 or 1 when it is divided by 3 (e.g consider remainders of the square of an even or odd number when divided by 3). Therefore, $p_1^2 + \dots + p_{17}^2$ is not a square, so we have shown that $p_1 \leq 3$.

   If $p_1 = 2$, then $p_{17}^2 - p_{16}^2$ is an even number so it is divisible by $p_1 = 2$. If $p_1 = 3$, then as before $p_{16} \equiv p_{17} = 1 \pmod 3$. Thus, $p_{17}^2 - p_{16}^2 = 0 \pmod 3$ which concludes the proof.

3. We consider the general case of n knights, $k_1, k_2, \dots k_n$, where $n \geq 3$. Let $s_1, s_2, \dots, s_n$ be the initial seats where $k_1, k_2, \dots, k_n$ sits in order, and let $a = \lfloor n/2 \rfloor$ be the greatest integer less than or equal to $n/2$. We split the knights into the two groups $K_1 =$

$\{k_1, k_2, \ldots, k_a\}$ and $K_2\{k_{a+1}, k_{a+2}, \ldots, k_n\}$, then we can change the anti-clockwise ordering of the seated knights into an clockwise ordering, by reversing the order of the knights in the set K_1 and K_2. To move k_1 to s_a, k_1 must swap position with k_2 then k_3 and so on. It takes $(a-1)$ swaps to move k_1 to the seat s_a. Similarly, it takes $(a-2)$ swaps to move k_2 into s_{a-1}, $(a-3)$ swaps to move k_3 to s_{a-2} and so on. Therefore, it takes $1+2+\ldots+(a-1)$ swaps to reverse the order of the set K_1. Similarly, it takes $1+2+\ldots+(n-a-1)$ swaps to reverse the order of the set K_2. In summary, then number of swaps required is

$$[1+2+\ldots+(a-1)]+[1+2+\ldots+(n-a-1)] = \sum_{r=2}^{n-1} \left\lfloor \frac{r}{2} \right\rfloor$$

Therefore, if $n = 12$ then the number of swaps required is 30, and if $n = 13$ then the number of swaps required is 36. All is left to do is to show that the number a we picked initially does indeed produce the minimum number of required swaps.