MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 14 Solutions, September 10, 2019

1. Suppose we take four socks and none of them match, then the next sock we draw must form a pair with one of the socks we have taken. Now to get another pair of socks, in the worst case we have to take two more socks. Therefore we have to take at least $5 + 2 \times 4 = 13$ socks from the drawer to guarantee that five of the pairs are matching.

2. For a number to be divisible by 12 it must be divisible by both 3 and 4. Since the sum of the digits of the 5-digit numbers we are forming is always 15, we need not check for divisibility by 3. If a number is divisible by 4, then its last two digits are divisible by four. Thus the possibilities for the last two digits of the number are $12, 24, 32$ and 52. For each choice of the last two digits, there are 3! ways to rearrange the first three digits, hence there are $3! \times 4 = 24$ ways to form a multiple of 12.

3. Label the wrestlers from 1 to 100 according to their strength in increasing order. In the first round, each wrestler labelled with an even number fights a wrestler with a label one less than theirs. Then all wrestlers labelled with an odd number are losers. In the second round, each odd number wrestler fights a wrestlers with a label one less than theirs with the exception of 1 and 100; they will fight each other. Then all even wrestlers except 100 are losers. Hence only one wrestler wins a prize.

4. (a) The number 33! has
   - 16 multiples of 2;
   - 8 multiples of 4;
   - 4 multiples of 8;
   - 2 multiples of 16; and
   - 1 multiple of 32

   The answer is 31.

   (b) In this case, we want to know how many powers of 10 are contained in 2019!. When considering $n!$, multiples of 2 easily outnumber multiples of 5, so this is tantamount to counting the number of multiples of 5. Applying the method of part (a), there are
   - 403 multiples of 5;
   - 80 multiples of 25;

- 16 multiples of 125; and
- 3 multiples of 625

So the answer is $403 + 80 + 16 + 3 = 502$.

5. Observe that $y = 0, x = 1, p = 2$ and $y = 1, x = 1, p = 5$ are two possible solutions. We show that there is no other. Assume $y > 0$, and write $p^x = y^4 + 4 = y^4 + 4y^2 - 4y^2 + 4 = (y^2 - 2y + 2)(y^2 + 2y + 2)$. Then $y^2 - 2y + 2 < y^2 + 2y + 2$, which implies $y^2 + 2y + 2$ is divisible by $y^2 - 2y + 2$ because their product is a power of the prime number p. On the other hand, we have $y^2 + 2y + 2 = (y^2 - 2y + 2) + 4y$; i.e $y^2 + 2y + 2$ has remainder $4y$ when it is divided by $y^2 - 2y + 2$. But since $y^2 + 2y + 2$ is divisible by $y^2 - 2y + 2$, we can conclude that $4y$ is either zero or $4y \geq y^2 - 2y + 2$. The former is impossible because $y > 0$ by assumption, and the latter implies $1 \leq y \leq 5$. We can easily check that there is no solution to the $p^x = y^4 + 4$ when $y = 2, 3, 4$ or 5.

6. Arrange the weights in increasing order, $a_1 < a_2 < \ldots < a_{11}$. Since the difference between any two weights is at least 1, we have $a_n \geq a_m + (n - m)$ for $m < n$. This implies,

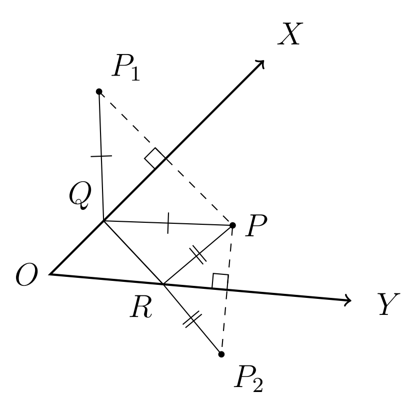
   $$a_7 + a_8 + a_9 + a_{10} + a_{11} \geq (a_2 + 5) + (a_3 + 5) + \ldots + (a_6 + 5) = a_2 + a_3 + a_4 + a_5 + a_6 + 25.$$

   Additionally, according to the given information we have

   $$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 > a_7 + a_8 + a_9 + a_{10} + a_{11}.$$

   Combining these two inequalities, we conclude that $a_1 > 25$, but then $a_{11} \geq a_1 + 10 > 35$.

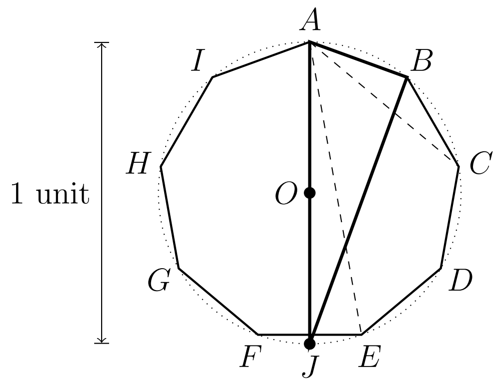
7. Let P_1 and P_2 be the reflections of the point P in OX and OY respectively.



   Then $P_1Q = PQ$ and $P_2Q = PQ$, and so the length of the path P_1QRP_2 is equal to the perimeter of $\triangle PQR$. Now, the points P_1 and P_2 are completely determined once the point P is known, so the only way to minimise the perimeter of the triangle is to make P_1QRP_2 a straight line.

Senior Questions

1. Without loss of generality, we can inscribe the regular nonagon in a circle centered at O with diameter 1 unit. Let J be the point on the circle opposite A, so that AOJ is a diameter of the circle. Then AJ is one unit.



Now, consider $\triangle ABJ$. As AJ is a diameter of the circle, $\angle JBA = \frac{\pi}{2}$. Furthermore, since $ACDEFGHI$ is a regular nonagon, $\angle AOB = \frac{2\pi}{9}$, which implies that $\angle BJA = \frac{\pi}{9}$. Thus $AB = \sin\frac{\pi}{9}$. By a similar argument, we can show that $AC = \sin\frac{2\pi}{9}$ and $AE = \sin\frac{4\pi}{9}$. Thus if we can show that

$$\sin\frac{4\pi}{9} - \sin\frac{2\pi}{9} = \sin\frac{\pi}{9},$$

then the desired result is proved.

By the sums to products identity, $\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$\begin{aligned}\sin\frac{4\pi}{9} - \sin\frac{2\pi}{9} &= 2\cos\left(\frac{4\pi+2\pi}{18}\right)\sin\left(\frac{4\pi-2\pi}{18}\right)\\ &= 2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{9}\right)\\ &= 2\times\frac{1}{2}\times\sin\left(\frac{\pi}{9}\right)\\ &= \sin\frac{2\pi}{9},\end{aligned}$$

as required.

2. Let the numbers be a, b, c, d and e increasing order.

   (a) Suppose the it is possible to form a triangle with sides length equal to these numbers, then $a + b > e$. Hence

   $$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + cd + de + (a + b)e,$$

   which is a contradiction.

   (b) Let us consider the following cases:

i. $b + c \leq d$. Then each of six triples in which two numbers are from the set $\{a, b, c\}$ and the third number is from the set $\{d, e, \}$ does not form a triangle.

ii. $c + d \leq e$. Then each of six triples which includes e does not form a triangle.

iii. $b + d \leq e$ and $a + b \leq d$. Then each of six triples $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, e\}, \{a, d, e\}$ $\{b, c, e\}$, $\{b, d, e\}$ does not form a triangle.
Suppose that neither of above cases takes place, that is, $b + c > d$, $c + d > e$ and at least one of inequalities $b + d > e$ and $a + b > d$ holds. We shall show that this is impossible.

iv. If $b + c > d$, $b + d > e$ then

$$a2 + b2 + c2 + d2 + e2 < ab + bc + ce + (b + c)d + (b + d)e.$$

Contradiction.

v. If $c + d > e$, $a + b > d$ then

$$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + cd + (a + d)d + (c + d)e.$$

Contradiction.