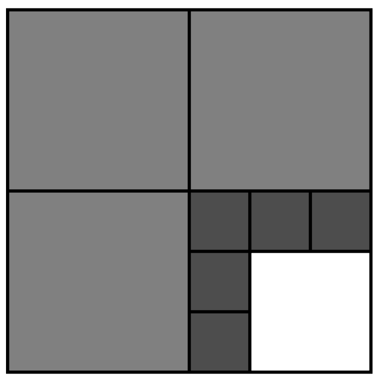
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 16 Solutions, September 17, 2019

1. Clearly $-a$ is the smallest number, as it is the only one that is negative. Note that, if $0 < a < 1$, multiplying by a makes the result *smaller*. Thus the correct order is: $-a$, a^3, a^2, a, $\sqrt{a}$.

2. Yes. See the diagram below



3. We can find the answer without solving for m and n explicitly.

$$\begin{aligned} m+n &= 11 \\ (m+n)^2 &= 121 \\ m^2+2mn+n^2 &= 121 \\ \therefore mn &= \frac{1}{2}(121-(m^2+n^2)) = 11 \end{aligned}$$

Now

$$\begin{aligned} (m+n)^3 &= 1331 \\ m^3+3m^2n+3mn^2+n^3 &= 1331 \\ m^3+n^3 &= 1331-3mn(m+n) \\ &= 968 \end{aligned}$$

4. If we consider the given equation as a quadratic in x, then, from the quadratic formula

$$x = \frac{8 \pm \sqrt{64+4004y^2}}{2}.$$

Since $x > 0$, we can simplify this as

$$\begin{aligned} x &= \frac{8 + \sqrt{64 + 4004y^2}}{2} \\ &= 4 + \sqrt{16 + 1001y^2} \end{aligned}$$

Note that if we use this formula, then x increases as y increases. Thus we want the smallest positive integer value of y that makes $\Delta = 16 + 1001y^2$ a perfect square, and then x will also be a positive integer.

y	Δ	x
1	1017	–
2	4020	–
3	$9025 = 95^2$	99

Thus the smallest value of $x + y$ is 102.

5. There is a simple algorithm for writing a number in another base, which I will use here. For the first step, we write the number using integer division (that is, in quotient and remainder form) with the given base. In the second step, we repeat this process with the quotient from the previous line. We continue in this fashion until we get a quotient of zero. Then the number is represented in the new base by the remainders written in reverse order.

$$\begin{aligned} 2000 &= -1000 \times (-2) + 0 \\ -1000 &= 500 \times (-2) + 0 \\ 500 &= -250 \times (-2) + 0 \\ -250 &= 125 \times (-2) + 0 \\ 125 &= -62 \times (-2) + 1 \\ -62 &= 31 \times (-2) + 0 \\ 31 &= -15 \times (-2) + 1 \\ -15 &= 8 \times (-2) + 1 \\ 8 &= -4 \times (-2) + 0 \\ -4 &= 2 \times (-2) + 0 \\ 2 &= -1 \times (-2) + 0 \\ -1 &= 1 \times (-2) + 1 \\ 1 &= 0 \times (-2) + 1 \end{aligned}$$

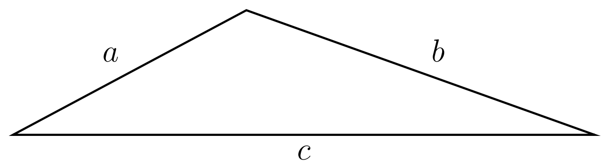
So we can see that $2000 = (1100011010000)_{-2}$.

If you want to double-check that this algorithm also works when we use negative numbers,[1] then

$$\begin{aligned}(-2)^{12} + (-2)^{11} + (-2)^{7} + (-2)^{6} + (-2)^{4} &= 4096 - 2048 - 128 + 64 + 16 \\ &= 2048 - 64 + 16 \\ &= 2000\end{aligned}$$

Thus there are 5 non-zero digits.

6. Let the lengths of the sides be a, b, and c, where $a < b < c$.



Firstly, note that $a \neq 2$. Since b and c are distinct primes, $c \geq b + 2$, which would make the triangle degenerate or impossible if $a = 2$.

Similarly, if $a = 3$, then by the triangle in equality, b and c are twin primes. However, after testing the values of some small twin primes, we find that the perimeter is not prime.

So let's consider $a = 5$. We soon find that $a = 5$, $b = 7$ and $c = 11$ works. Thus the smallest perimeter is 23.

Senior Questions

1. We need k and n such that,

$$2^{k+1} + \ldots + 2^{k+10} = 1 + 2 + \ldots + n$$

Now the LHS is a geometric series, and the RHS is an arithmetic series. This simplifies to $2^{k+2}(2^{10} - 1) = n(n + 1)$, which holds for $n = 2^{10} - 1$ and $k = 8$.

2. Let

$$a_n = \sqrt[n]{\frac{(2n)!}{n!n^n}}.$$

If we take the natural log on both sides of the above equation, then

$$\begin{aligned}\ln(a_n) &= \frac{1}{n}\left(\ln\left(\frac{(n+1)\times(n+2)\times\ldots\times(n+n)}{n^n}\right)\right) \\ &= \frac{1}{n}\left(\ln\left(\frac{n+1}{n}\right) + \ln\left(\frac{n+2}{n}\right) + \ldots + \ln\left(\frac{n+n}{n}\right)\right) \\ &= \sum_{k=1}^{n} \frac{1}{n} \ln\left(1 + \frac{k}{n}\right).\end{aligned}$$

[1] I learned this algorithm in the context of a positive base, so I must admit that I was slightly sceptical about whether it would work with negative integers. However, it seems that it does, as long as we allow negative quotients (but positive remainders) in the algorithm.

Notice that the RHS of the last displayed equation is a Riemann sum of $\ln(x)$ for $x \in [1, 2]$. As n gets very large, the Riemann sum for the approximation of $\ln(x)$ becomes the exact integral

$$\int_1^2 \ln(x)\, dx = 2\ln 2 - 1,$$

where we have solved the integral using integration by parts. Hence $\lim_{n\to\infty} a_n = \exp(2\ln 2 - 1)$ (to obtain the conclusion, we have used the fact that $\ln(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} \ln a_n$, why is this true?).

3. Lets label the two known equations:

$$x^4y^5 + y^4x^5 = 810, \tag{1}$$

and

$$x^3y^6 + y^3x^6 = 945. \tag{2}$$

Consider $(x+y)^3 = x^3 + y^3 + 3(x^2y + y^2x)$. From (1), we have $x^2y + y^2x = \frac{810}{(xy)^3}$ and $x + y = \frac{810}{(xy)^4}$. Therefore

$$\begin{aligned}(x+y)^3 &= x^3 + y^3 + 3(x^2y + y^2x)\\ \frac{810^3}{(xy)^{12}} &= x^3 + y^3 + \frac{2430}{(xy)^3}.\end{aligned} \tag{3}$$

Also, from (2) $x^3 + y^3 = \frac{945}{(xy)^3}$, therefore the second line of (3) becomes

$$\frac{810^3}{(xy)^{12}} = \frac{945}{(xy)^3} + \frac{2430}{(xy)^3},$$

which implies $(xy)^9 = \frac{810^3}{945+2430}$. Therefore, $(xy)^3 = 54$ and $x^3 + y^3 = 17.5$.
We can now easily evaluate $2x^3 + (xy)^3 + 2y^3 = 89$.