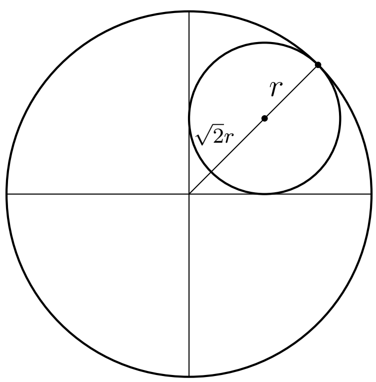
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 17 Solutions, September 30, 2019

1. If a number is divisible by 8, then its last three digits are a multiple of 8. This gives us $b = 4$. If a number is divisible by 9, then the sum of its digits is also divisible by 9. This gives us $a = 9$. Thus $a + b = 9 + 4 = 13$.

2. Suppose that the smaller circle has radius r. Then the distance between the centres of the two circles is $\sqrt{2}r$. Thus the radius of the big circle is $(1 + \sqrt{2})r$.



So the ratio is

$$\pi r^2 : \frac{\pi(1+\sqrt{2})^2 r^2}{4},$$

which simplifies to

$$4 : (3 + 2\sqrt{2}).$$

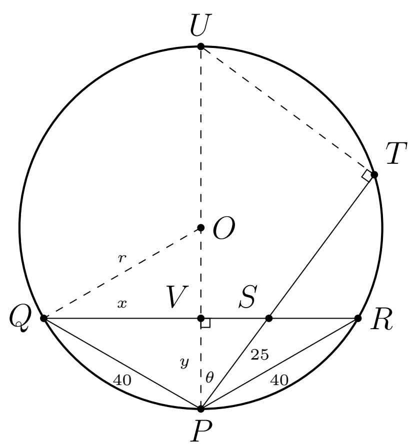
3. The edge of the cube is 16 units; the length of the diagonal on any of the faces is $16\sqrt{2}$ units; and the length of a diagonal through the middle of the cube is $16\sqrt{3}$. If we place 8 points on the cube spacing them as far as possible from each other (that is, on the 8 vertices), then the minimum distance between points is 16. However, if we then place a ninth point at the centre of the cube, then the distance from this point to any other point is $8\sqrt{3} \approx 13.84\ldots < 14$. So the answer is nine.

4. Let's calculate a few terms, and see if we can find a pattern.

$$\begin{aligned}
x_1 &= \sqrt{2} \\
x_2 &= \sqrt{3} \\
x_3 &= x_2 - x_3 = \sqrt{3} - \sqrt{2} \\
x_4 &= x_3 - x_2 = (\sqrt{3} - \sqrt{2}) - \sqrt{3} = -\sqrt{2} \\
x_5 &= x_4 - x_3 = -\sqrt{2} - (\sqrt{3} - \sqrt{2}) = -\sqrt{3} \\
x_6 &= x_5 - x_4 = -\sqrt{3} + \sqrt{2} \\
x_7 &= x_6 - x_5 = (-\sqrt{3} + \sqrt{2}) + \sqrt{3} = \sqrt{2} \\
x_8 &= x_7 - x_6 = \sqrt{2} - (-\sqrt{3} + \sqrt{2}) = \sqrt{3}
\end{aligned}$$

Since $x_7 = x_1$ and $x_8 = x_2$, we can see that the same pattern will keep repeating with period 6. Now $2019 = 336 \times 6 + 3$, so $x_{2019} = x_3 = \sqrt{3} - \sqrt{2}$.

5. Suppose that the circle has radius r and centre O. Let PU be a diameter of the circle, and let QR and PU intersect at the point V. Let $QV = x$, $VP = y$, and $\angle UPT = \theta$, as shown in the diagram.



Using basic properties of circle geometry, we can show that $UP \perp QR$ and that $\triangle UPT$ is similar to SPV. Thus

$$\begin{aligned}
\cos\theta &= \frac{y}{25} = \frac{PT}{UP} \\
PT &= \frac{2ry}{25}
\end{aligned}$$

Now, consider $\triangle QVP$. As we know, $\triangle QVP$ is right-angled, which implies that $x^2 + y^2 = 40^2$. Similarly, from $\triangle OQV$, we can see that

$$\begin{aligned}
x^2 + (r - y)^2 &= r^2 \\
x^2 + r^2 - 2ry + y^2 &= r^2 \\
\therefore 2ry &= x^2 + y^2 \\
&= 40^2
\end{aligned}$$

Thus $PT = \frac{40^2}{25} = 64$.

Senior Questions

1. Rearranging the given equation,

$$\begin{aligned}
3x^2 - 8y^2 + 3x^2y^2 &= 2008 \\
3x^2(1+y^2) &= 8(251+y^2) \\
3x^2 &= \frac{8(250+1+y^2)}{1+y^2} \\
3x^2 &= 8\left(1+\frac{250}{1+y^2}\right)
\end{aligned}$$

Since the RHS is an integer, this means that $1+y^2$ is a factor of 250. The factors of 250 are 1, 2, 5, 10, 25, 50, 125, 250, which gives possible values of y of 1, 2, 3, and 7. However, the only one of these values that gives a multiple of 3 is $y = 7$, which gives $x = 4$.

2. Since f is a polynomial, we can write it as

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = \sum_{k=0}^{n} a_kx^k,$$

where $a_0, \ldots, a_n$ are non-negative integers. Furthermore,

$$f(1) = a_0 + a_1 + a_2 + \ldots + a_n = \sum_{k=0}^{n} a_k.$$

Since $f(1) = 6$, this tells us that at most 6 of the a_k are non-zero, and also that no coefficient is larger than 6. This means that we can find the coefficients of f by writing 3438 in base 7, which is the same as writing 3438 in terms of integer multiples of powers of 7.

Using the change of base algorithm we discussed in the solutions to Problem Sheet 16,

$$\begin{aligned}
3438 =& 491 \times 7 + 1 \\
491 =& 70 \times 7 + 1 \\
70 =& 10 \times 7 + 0 \\
10 =& 1 \times 7 + 3 \\
1 =& 0 \times 7 + 1.
\end{aligned}$$

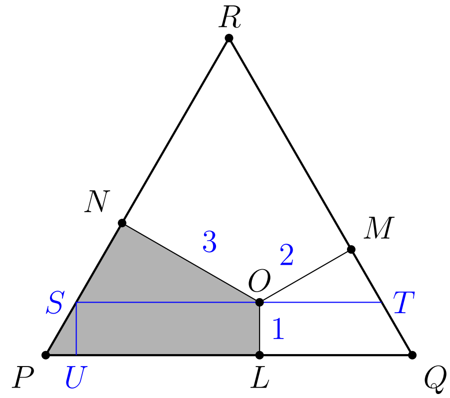
So

$$\begin{aligned}
3438 &= (13011)_7 \\
\implies f(7) &= 1 \times 7^4 + 3 \times 7^3 + 0 \times 7^2 + 1 \times 7^1 + 1 \times 7^0 \\
\therefore f(x) &= x^4 + 3x^3 + x + 1.
\end{aligned}$$

Evaluating f at $x = 3$, we obtain 166.

3. We rescale the triangle so that $OL = 1$, $OM = 2$, and $ON = 3$.

Firstly, we will find the area of the shaded region. Draw the line ST, parallel to PQ and passing through O, and let SU be the perpendicular from S to PQ. Then $\triangle RST$ is also an equilateral triangle, and using some basic trigonometry we can calculate that $SO = 2\sqrt{3}$, $SN = \sqrt{3}$, $PU = 1/\sqrt{3}$.



Then $SOLU$ has area $2\sqrt{3}$; $\triangle SPU$ has area $\frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$; and $\triangle NSO$ has area $\frac{3\sqrt{3}}{2}$. Thus

$$\text{Area of } LONP = 2\sqrt{3} + \frac{\sqrt{3}}{6} + \frac{3\sqrt{3}}{2} = \frac{11\sqrt{3}}{3}.$$

We will now calculate the area of $\triangle PQR$. Recall from Question 5 on Problem Sheet 10, 2018, that the sum of the perpendiculars from any interior point of an equilateral triangle is equal to the altitude of the triangle. Thus we know that $\triangle PQR$ has altitude 6, side length $4\sqrt{3}$, and hence area $12\sqrt{3}$.

Thus

$$\frac{a}{b} = \frac{11\sqrt{3}/3}{12\sqrt{3}} = \frac{11}{36},$$

and so $a + b = 47$.