1. It is not very elegant, but the quickest way to solve this problem is probably brute force. That is, write out the first few powers of 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048. We notice that $2048 - 32 = 2016$. Consequently $a = 11$ and $b = 5$, so $a + b = 16$.

2. Let $O$ be the midpoint of $NM$, extend the line $AB$ so that it intercepts $KN$ at the point $P$; see below. Since $NM$ and $PL$ are parallel and $O$ is the midpoint of $NM$, $A$ is the midpoint of $PL$ (this is a special case of the intercept theorem [http://en.wikipedia.org/wiki/Intercept_theorem](http://en.wikipedia.org/wiki/Intercept_theorem)). Therefore the triangles $PNA$ and $ANL$ are congruent to each other, hence $\angle PNA = \angle ANL$.

3. We can write $n$ as $n = 3^{a_5b_7c} \times N$, where the number $N$ has no factors of 3, 5 or 7. Then $\frac{1}{3}n = 3^{a-1}5^{b-1}7^{c} \times N$, $\frac{1}{5}n = 3^{a}5^{b-1}7^{c} \times N$ and $\frac{1}{7}n = 3^{a}5^{b}7^{c-1} \times N$. Because we are looking minimal $N$, we may as well set $N = 1$. So for $\frac{1}{3}n$ to be a perfect cube, $\frac{1}{3}n$ to be a perfect fifth power and $\frac{1}{3}n$ to be a perfect seventh power, we must have $a - 1$ a multiple of 3 and $a$ itself a multiple of 5 and 7 (i.e., a multiple of 35). The smallest the smallest such $a$ is 70. To find $n$, repeat this argument to obtain $b$ and $c$.

4. We have
\[
k^3 - 1 = (k - 1)(k^2 + k + 1) = (k - 1)(k(k + 1) + 1)
\]
and
\[
k^3 + 1 = (k + 1)(k^2 - k + 1) = (k + 1)(k(k - 1) + 1).
\]
Therefore the numerator of the given product contains the factors $1, 2, 3, \ldots, n - 1$ and the denominator contains $3, 4, 5, \ldots, n + 1$. Most of these cancel and we are left with
\[ \frac{2}{n(n+1)}. \] The numerator also contains factors \(2 \times 3 + 1, 3 \times 4 + 1, \ldots, n(n+1) + 1\), and the denominator \(1 \times 2 + 1, 2 \times 3 + 1, \ldots, (n+1) + 1\); again most cancel and there remains \((n(n+1) + 1)/(1 \times 2 + 1)\). Combining all these results gives

\[
\frac{2^3 - 1}{2^3 + 1} \frac{3^3 - 1}{3^3 + 1} \cdots \frac{n^3 - 1}{n^3 + 1} = \frac{2}{n(n+1)} \frac{n(n+1) + 1}{1 \times 2 + 1} = \frac{2n^2 + n + 1}{3n^2 + n}.
\]

5. Let \(M_1\) and \(M_2\) be the two mathematicians. We can plot the arrival time of \(M_1\) and \(M_2\) on the \(x - y\) plane, with \(x\)-axis representing the arrival time of \(M_1\), and \(y\)-axis the arrival time of \(M_2\); see figure ???. Each mathematician stays in the tea room for exactly \(m\) minutes, so we know that if \(M_1\) arrives first (say at 9 a.m.) then \(M_2\) will run into \(M_1\) in the cafeteria if \(M_2\)’s arrival time is within \(m\) minutes of \(M_1\); This is represented by the \(m \times m\) square box in the bottom left of the plot. Over the break of 60 minutes, we get a shaded region as shown in figure ???. The probability that either mathematician arrives while the other is in the cafeteria is 40%, thus the non-shaded region is 60% of the total area of the big square. So we have

\[
\frac{(60 - m)^2}{60^2} = 0.6
\]

\[
m = 60 - 12\sqrt{15},
\]

therefore, \(a + b + c = 87\).

6. Let \(f(n)\) be the number of ways we can choose these \(n\) integers. We can try to workout what \(f(n + 1)\) is; that is the number of ways to choose \(x_1, x_2, \ldots, x_n, x_{n+1}\) such that each is 0, 1 or 2 and their sum even.

Suppose we have \(n\) integers, \(x_1, \ldots x_n\) from the list 0, 1, 2 such that their sum is even. We know there is \(f(n)\) ways to choose these \(n\) numbers, and we can either pick \(x_{n+1}\) to be 0 or 2 so that the sum of \(x_1, \ldots, x_{n+1}\) is even; the total number of ways we can pick these \(n + 1\) integers is \(2f(n)\).

On the other hand, if the initial \(n\) integers, \(x_1, \ldots x_n\) from the list 0, 1, 2 is odd, then there is \(3^n - f(n)\) ways to choose these \(n\) numbers, and we can only pick \(x_{n+1} = 1\) so that the sum of \(x_1, \ldots x_{n+1}\) is even; the total number of ways we can pick these \(n + 1\) integers is \(3^n - f(f)\)

Combining both cases, we have the recursive relation \(f(n + 1) = 3^n + f(n)\). Since it is straightforward to workout \(f(1) = 2\), we can find \(f(n)\).
Senior Questions

1. Given that $a$, $b$, and $c$ are positive integers, solve

(a) If $a > b$, then dividing both sides by $a!$, we have

$$b! = \frac{b!}{a!} + 1,$$

the LHS of the above equation is an integer, while the RHS is not; we have a contradiction on the condition $a > b$. We can apply the same arguments to get $a \not< b$, so that $a = b$. The only solution is then $a = b = 2$.

(b) Notice this equation is symmetric in $a$ and $b$, so we can assume without loss of generality $a \geq b$. Dividing through by $b!$, then

$$a! = \frac{a!}{b!} + 1 + \frac{2^c}{b!}.$$  \hspace{1cm} (1)

The LHS of equation (1) is an integer and $a!/b!$ is an integer, therefore $2^c/b!$ must be an integer, this implies $b$ is either 1 or 2. Also, the RHS of (1) is the sum of 3 integers, so $a!$ must contain a factor of 3; $a \geq 3$.

If $b = 1$ then $a! = a! + 1 + 2^c$, which implies $2^c + 1 = 0$; there is no solution for $c$, so $b \neq 1$. Therefore $b = 2$.

If $a > 3$, then $a!/2$ is even, so $2^{c-1} = 1$. But then we get $a!/2 = 2$, which has no solution for $a$.

Therefore, we conclude that $a = 3$ and $b = 2$, therefore $c = 2$.

(c)

2. (a) The inequality holds for $n = 3$. Assume $n! > (n - 2)(1! + 2! + \ldots (n - 1)!)$ and note that $2(n - 2) \geq n - 1$ for $n \geq 3$, therefore

$$(n + 1)! = (n - 1)n! + 2n!$$

$$> (n - 1)n! + 2(n - 2)(1! + 2! + \ldots (n - 1)!)$$

$$\geq (n - 1)(1! + 2! + \ldots + n!),$$

so the inequality holds for all $n$ by standard induction arguments.

(b) $(n + 1)! < n(1! + 2! + \ldots + n!)$ because

$$(n + 1)! = (n + 1)n!$$

$$= nn! + n!$$

$$= n(n! + (n - 1)!)$$

$$< n(1! + 2! + \ldots + n!).$$

Therefore, combining with the result of (a),

$$n < \frac{(n + 1)!}{1! + 2! + \ldots + n!} < n + 1.$$  

So $(n + 1)!$ divided by $1! + 2! + \ldots n!$ is a number that is strictly between $n$ and $n + 1$; $1! + 2! + \ldots n!$ does not divide $(n + 1)!$. 

3