1. Note that $x$ has greater magnitude than $y$. Firstly, let’s concentrate on positive solutions to the equation. If $2019 = x^2 - y^2$, then $2019 = (x + y)(x - y)$. The factors of 2019 are 1, 3, 673, and 2019. So

<table>
<thead>
<tr>
<th>$(x - y)$</th>
<th>$(x + y)$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2019</td>
<td>1010</td>
<td>1009</td>
</tr>
<tr>
<td>3</td>
<td>673</td>
<td>338</td>
<td>335</td>
</tr>
</tbody>
</table>

Thus the solutions are $(1010, \pm 1009), (-1010, \pm 1009), (338, \pm 335)$ and $(-338, \pm 335)$, so there are eight solutions altogether.

2. Let $O$ be the centre of the incircle, let the radius of the incircle be $r$ and let $D$, $E$ and $F$ be the points of tangency between the incircle and the triangle as shown below.

![Diagram of triangle with incircle](image)

Since $OD$, $OE$, and $OF$ are radii to tangents, $\angle BFO = \angle CEO = \angle ODB = 90^\circ$. Thus $AFOE$ is a square with side length $r$. Hence $AE = AF = r$, $EC = b - r$ and $FB = c - r$. Furthermore, by RHS, $\triangle EOC \equiv \triangle DOC$ and thus $DC = b - r$. Similarly, $BD = c - r$. Thus

\[
a = (b - r) + (c - r)
\]

\[
r = \frac{1}{2}(b + c - a)
\]

\[\text{Some problems from UNSW’s publication Parabola, and the Tournament of Towns in Toronto}\]
3. A neat trick is to express $N$ as

$$\frac{333\ldots333}{61\times9's} = \frac{3}{9} \left(\frac{999\ldots999}{61\times9's}\right) = \frac{1}{3}(10^{61} - 1).$$

Similarly, $M = \frac{666\ldots666}{62\times6's} = \frac{2}{3}(10^{62} - 1)$. Now

$$N \times M = \frac{2}{9}(10^{61} - 1)(10^{62} - 1)$$

$$= \frac{2}{9}(10^{61} - 1) \times 10^{62} - \frac{2}{9}(10^{61} - 1)$$

$$= \frac{222\ldots222}{60\times2's} \frac{000\ldots000}{62\times0's} - \frac{222\ldots222}{60\times2's}$$

$$= \frac{222\ldots222}{60\times2's} \frac{19777\ldots777}{60\times7's}.$$

4. In modular arithmetic, if $a \equiv b \mod(n)$, then $a^x \equiv b^x \mod(n)$. Thus we can see that

$$a \equiv 1 \mod(a - 1)$$

$$a^x \equiv 1^x \mod(a - 1)$$

$$1 \mod(a - 1)$$

Similarly,

$$a \equiv a \mod(a + 1)$$

$$\equiv (-1) \mod(a + 1)$$

$$a^x \equiv (-1)^x \mod(a + 1)$$

$$(-1)^x \mod(a + 1) \equiv -1 \mod(a + 1)$$

$$\equiv a$$

Thus $r_1 + r_2 = a + 1$.

5. We can write $x = n + d$, where $n$ is the integral part of $x$ and $d$ the decimal part. Then $[2x] + [4x] + [6x] + [8x] = 20n + [2d] + [4d] + [6d] + [8d]$. We scan over the range of $d$; that is $0 < d < 1$ to see what positive integer under 1001 can be expressed in the form of $[2x] + [4x] + [6x] + [8x]$. For example

<table>
<thead>
<tr>
<th>$[2x]$</th>
<th>$+ [4x]$</th>
<th>$+ [6x]$</th>
<th>$+ [8x]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
If we continue with the above calculations, the results are the numbers ending in 3, 7, 8 or 9 can not be expressed in the form \([2x] + [4x] + [6x] + [8x]\). This means that, for \(n = 0\), we have 0 (in this case we can’t actually count this one, as we are looking at positive integers), 1, 2, 4, 5 and 6. For \(n = 1\), we have 20, 21, 22, 24, 25 and 26 (6 possibilities). For \(n = 2\), we have 40, 41, 42, 24, 45 and 46, and so on. Since we are also counting 1000 itself, there are a total of 300 numbers that can be written this way.

6. Let \(d\) be the number of kilometres travelled before the tyre switch is made. Then \(\frac{d}{x}\) is the proportion of wear on the front tyre before the switch, hence they will travel a further \((1 - \frac{d}{x})\) \(y\) kilometres before the tyres are retired. So the total distance travelled by the front tyre is \(d + (1 - \frac{d}{x})\) \(y\). Similarly, the total distance travelled by the rear tyre is \(d + (1 - \frac{d}{y})\) \(x\).

Suppose the claim of the advertisement is true, then we must have the following system of inequalities

\[
d + \left(1 - \frac{d}{x}\right) y \geq \frac{x+y}{2},
\]

\[
d + \left(1 - \frac{d}{y}\right) x \geq \frac{x+y}{2}.
\]

Rearranging this gives

\[
d\left(1 - \frac{y}{x}\right) \geq \frac{x-y}{2},
\]

\[
d\left(1 - \frac{x}{y}\right) \geq \frac{y-x}{2},
\]

then using the assumption that \(x < y\), we have

\[
d \leq \frac{x-y}{2} \times \left(1 - \frac{y}{x}\right)^{-1} = \frac{x}{2}
\]

\[
d \geq \frac{x-y}{2} \times \left(1 - \frac{x}{y}\right)^{-1} = \frac{y}{2}.
\]

The last system of inequality does not hold because \(x < y\), so we have a contradiction to the advertisement’s claim.
Senior Questions

1. Since $\alpha > 0$, $(\alpha + \frac{1}{\alpha})^2 = \alpha^2 + \frac{1}{\alpha^2} + 2 \geq 2$. Similarly, $(\beta + \frac{1}{\beta})^2 \geq 2$. Therefore, if $r_1$ and $r_2$ are the roots of $f$ (assuming $r_1 \geq r_2$ wlog), then $r_1 \geq 2$ and $r_2 < 0$, so that $r_1r_2 = c - 3 < 0$, which implies $c < 3$.

To get the lower bound on $c$, we use the quadratic formula $2 \leq r_1 = (c + 1) + \sqrt{(c+1)^2 - 4(c-3)}$. Solving gives $-2 \leq c$.

2. Square both sides of the equation $\sqrt{a} - b = \sqrt{c}$ and rearranging gives

$$\sqrt{c} = \frac{a - b^2 - c}{2b}.$$ 

Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x/y$, where $x$ and $y$ are integers with greatest common multiplier one. Then $c = x^2/y^2$, and greatest common multiplier between $x^2$ and $y^2$ is one. Since $c$ is an integer, $x^2$ must be divisible by $y^2$, which can only happen if $y^2 = 1$, because the greatest common multiplier between $x^2$ and $y^2$ is one. Hence $c = x^2$, so that $c$ is a perfect square.

If $c$ is a perfect square, then the equation $\sqrt{a} - b = \sqrt{c}$ implies that $a$ is also a perfect square.

3. Use the method of reflection. Reflect the point $B$ in the line that represents the river bank. This is shown as $B'$ in the diagram below. Then the shortest distance from $A$ to $B'$ is clearly a straight line. We can use Pythagoras’ theorem to show that this is 15 km.