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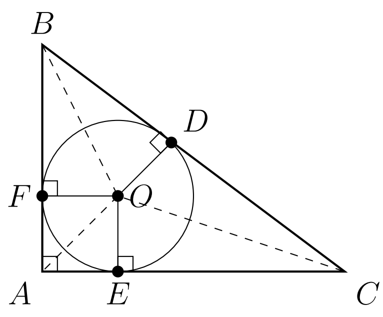
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 25, 2019[1]

1. Note that x has greater magnitude than y. Firstly, let's concentrate on positive solutions to the equation. If $2019 = x^2 - y^2$, then $2019 = (x+y)(x-y)$. The factors of 2019 are 1, 3, 673, and 2019. So

$(x-y)$	$(x+y)$	x	y
1	2019	1010	1009
3	673	338	335

Thus the solutions are $(1010, \pm 1009)$, $(-1010, \pm 1009)$, $(338, \pm 335)$ and $(-338, \pm 335)$, so there are eight solutions altogether.

2. Let O be the centre of the incircle, let the radius of the incircle be r and let D, E and F be the points of tangency between the incircle and the triangle as shown below.



Since OD, OE, and OF are radii to tangents, $\angle BFO = \angle CEO = \angle ODB = 90°$. Thus $AFOE$ is a square with side length r. Hence $AE = AF = r$, $EC = b - r$ and $FB = c - r$. Furthermore, by RHS, $\triangle EOC \equiv \triangle DOC$ and thus $DC = b - r$. Similarly, $BD = c - r$. Thus

$$a = (b-r) + (c-r)$$
$$r = \frac{1}{2}(b + c - a)$$

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

3. A neat trick is to express N as

$$\underbrace{333\ldots333}_{61\times 3's} = \frac{3}{9}\left(\underbrace{999\ldots999}_{61\times 9's}\right) = \frac{1}{3}(10^{61}-1).$$

Similarly, $M = \underbrace{666\ldots666}_{62\times 6's} = \frac{2}{3}(10^{62}-1)$. Now

$$\begin{aligned} N\times M &= \frac{2}{9}(10^{61}-1)(10^{62}-1) \\ &= \frac{2}{9}(10^{61}-1)\times 10^{62} - \frac{2}{9}(10^{61}-1) \\ &= \underbrace{222\ldots222}_{60\times 2's}\underbrace{000\ldots000}_{62\times 0's} - \underbrace{222\ldots222}_{60\times 2's} \\ &= \underbrace{222\ldots222}_{60\times 2's}19\underbrace{777\ldots777}_{60\times 7's}8. \end{aligned}$$

4. In modular arithmetic, if $a \equiv b\text{mod}(n)$, then $a^x \equiv b^x\text{mod}(n)$. Thus we can see that

$$\begin{aligned} a &\equiv 1\text{mod}(a-1) \\ a^x &\equiv 1^x\text{mod}(a-1) \\ &1\text{mod}(a-1) \end{aligned}$$

Similarly,

$$\begin{aligned} a &\equiv a\text{mod}(a+1) \\ &\equiv (-1)\text{mod}(a+1) \\ a^x &\equiv (-1)^x\text{mod}(a+1) \end{aligned}$$

$$\begin{aligned} (-1)^x\text{mod}(a+1) &\equiv -1\text{mod}(a+1) \\ &\equiv a \end{aligned}$$

Thus $r_1 + r_2 = a + 1$.

5. We can write $x = n + d$, where n is the integral part of x and d the decimal part. Then $[2x] + [4x] + [6x] + [8x] = 20n + [2d] + [4d] + [6d] + [8d]$. We scan over the range of d; that is $0 < d < 1$ to see what positive integer under 1001 can be expressed in the form of $[2x] + [4x] + [6x] + [8x]$. For example

$[2x]$	+	$[4x]$	+	$[6x]$	+	$[8x]$		
0	+	0	+	0	+	1	$= 1,$	if $\frac{1}{8} \le d < \frac{1}{6}$.
0	+	0	+	1	+	1	$= 2,$	if $\frac{1}{6} \le d < \frac{1}{4}$.
0	+	1	+	1	+	2	$= 4,$	if $\frac{1}{4} \le d < \frac{1}{3}$.
0	+	1	+	2	+	2	$= 5,$	if $\frac{1}{3} \le d < \frac{3}{8}$.
0	+	1	+	2	+	3	$= 6,$	if $\frac{3}{8} \le d < \frac{1}{2}$.

If we continue with the above calculations, the results are the numbers ending in $3, 7, 8$ or 9 can not be expressed in the form $[2x] + [4x] + [6x] + [8x]$. This means that, for $n = 0$, we have 0 (in this case we can't actually count this one, as we are looking at positive integers), 1, 2, 4, 5 and 6. For $n = 1$, we have 20, 21, 22, 24, 25 and 26 (6 possibilities). For $n = 2$, we have 40, 41, 42, 24, 45 and 46, and so on. Since we are also counting 1000 itself, there are a total of 300 numbers that can be written this way.

6. Let d be the number of kilometres travelled before the tyre switch is made. Then $\frac{d}{x}$ is the proportion of wear on the front tyre before the switch, hence they will travel a further $\left(1 - \frac{d}{x}\right) y$ kilometres before the tyres are retired. So the total distance travelled by the font tyre is $d + \left(1 - \frac{d}{x}\right) y$. Similarly, the total distance travelled by the rear tyre is $d + \left(1 - \frac{d}{y}\right) x$.

   Suppose the claim of the advertisement is true, then we must have the following system of inequalities

$$\begin{aligned} d + \left(1 - \tfrac{d}{x}\right) y &\geq \tfrac{x+y}{2} \\ d + \left(1 - \tfrac{d}{y}\right) x &\geq \tfrac{x+y}{2} \end{aligned}.$$

   Rearranging this gives

$$\begin{aligned} d\left(1 - \tfrac{y}{x}\right) &\geq \tfrac{x-y}{2} \\ d\left(1 - \tfrac{x}{y}\right) &\geq \tfrac{y-x}{2}, \end{aligned}$$

   then using the assumption that $x < y$, we have

$$\begin{aligned} d &\leq \tfrac{x-y}{2} \times \left(1 - \tfrac{y}{x}\right)^{-1} = \tfrac{x}{2} \\ d &\geq \tfrac{x-y}{2} \times \left(1 - \tfrac{x}{y}\right)^{-1} = \tfrac{y}{2}. \end{aligned}$$

   The last system of inequality does not hold because $x < y$, so we have a contradiction to the advertisement's claim.

Senior Questions

1. Since $\alpha > 0$, $\left(\alpha + \frac{1}{\alpha}\right)^2 = \alpha^2 + \frac{1}{\alpha^2} + 2 \geq 2$. Similarly, $\left(\beta + \frac{1}{\beta}\right)^2 \geq 2$. Therefore, if r_1 and r_2 are the roots of f (assuming $r_1 \geq r_2$ wlog), then $r_1 \geq 2$ and $r_2 < 0$, so that $r_1 r_2 = c - 3 < 0$, which implies $c < 3$.

   To get the lower bound on c, we use the quadratic formula $2 \leq r_1 = (c+1) + \sqrt{(c+1)^2 - 4(c-3)}$. Solving gives $-2 \leq c$.

2. Square both sides of the equation $\sqrt{a} - b = \sqrt{c}$ and rearranging gives

$$\sqrt{c} = \frac{a - b^2 - c}{2b}.$$

   Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x/y$, where x and y are integers with greatest common multiplier one. Then $c = x^2/y^2$, and greatest common multiplier between x^2 and y^2 is one. Since c is an integer, x^2 must be divisible by y^2, which can only happen if $y^2 = 1$, because the greatest common multiplier between x^2 and y^2 is one. Hence $c = x^2$, so that c is a perfect square.

   If c is a perfect square, then the equation $\sqrt{a} - b = \sqrt{c}$ implies that a is also a perfect square.

3. Use the method of reflection. Reflect the point B in the line that represents the river bank. This is shown as B' in the diagram below. Then the shortest distance from A to B' is clearly a straight line. We can use Pythagoras' theorem to show that this is 15 km.