MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 9, July 8, 2019

1. Given a n-digit long number, if we fix the last digit (say let it be 1), then there are $(n-1)!$ ways to arrange the other $n-1$ digits (say $2, 3, \ldots, n$) to get a different n-digit long number. Hence, each of $1, 2, \ldots, n$ will appear $(n-1)!$ times in the last digit of the n-digit long number. Therefore the last digit of all combinations of n-digit long numbers contributes to the sum by an amount of $(n-1)! \times (1+2+3+\ldots+n)$.

   We can make a similar argument by fixing the second last digit, so that the second last digit of all combinations of n-digit long numbers contributes to the sum $(n-1)! \times (1+2+3+\ldots+n) \times 10$.

   Repeating this for all digits, the total sum is $(n-1)! \times (1+2+3+\ldots+n) \times (1+10+10^2+\ldots+10^n)$.

2. Note that $\sqrt[4]{2} \times \sqrt[8]{4} \times \sqrt[16]{8} \times \sqrt[32]{16} \times \sqrt[64]{32} \ldots = \sqrt[4]{2} \times \sqrt[8]{2^2} \times \sqrt[16]{2^3} \times \sqrt[32]{2^4} \times \sqrt[64]{2^5} \ldots$. Hence we can rewrite this product as

$$2^{\frac{1}{4}} \times 2^{\frac{2}{8}} \times 2^{\frac{3}{16}} \times 2^{\frac{4}{32}} \times 2^{\frac{5}{64}} \ldots = 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\frac{5}{64}} \ldots,$$

   which means that all we need to find is the infinite sum $x = 1/4 + 2/8 + 3/16 + 4/32 + 5/64 + \ldots$. The trick here is to compare x with $x/2$:

$$\begin{aligned} x - \frac{x}{2} &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots \\ \frac{x}{2} &= \frac{1/4}{1-1/2} \\ x &= 1. \end{aligned}$$

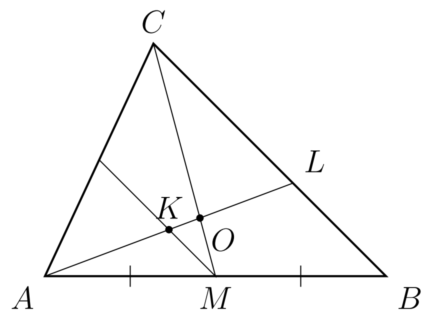
   We conclude that product of the surds is 2.

3. Let a be the greatest number written on a blackboard. Pick another integer, b, on the board, then $a \geq b$. Furthermore, there is an integer $n \geq 0$ such that $2^n \leq a < 2^{n+1}$, so that $2^n < a + b \leq 2a < 2^{n+2}$. Since $a + b$ must be a power of two, we must have $a + b = 2^{n+1}$. Because b is an arbitrary integer we picked and there is only one choice of n, we can conclude that $2^{n+1} - a$ is the only other integer on the board; there are only two numbers on the board.

4. (a) Since the lines KM and CB are parallel, $\angle KMO = \angle OCL$. Furthermore, $\angle COL$ and $\angle MOK$ are vertically opposite and hence equal. Thus $\triangle KMO$ is similar to $\triangle LCO$, and we have the formula

$$\frac{|KO|}{|KL| - |KO|} = \frac{|OM|}{|MC| - |OM|}.$$

(We need this for the second part.)



NOT TO SCALE

(b) Since M is the midpoint of AB and the lines KM and CB are parallel, by the midpoint theorem K is the midpoint of AL. Additionally, using the fact that $2|MC| = |AL|$, we have $|KL| = |MC|$. Now substituting $|KL| = |MC|$ into the formula from part (a), we obtain $|KO| = |OM|$. Therefore the triangles $\triangle KMO$ and $\triangle OLC$ are isosceles. Finally, using the condition $\angle OLC = 45°$ we have $\angle COL = 90°$.

5. Since the constant coefficient of $p(x)$ is 3, $abcd = 3$. Therefore,

$$\frac{abc}{d} = \frac{3}{d^2}, \qquad \frac{acd}{b} = \frac{3}{b^2}, \qquad \frac{abd}{c} = \frac{3}{c^2}, \qquad \frac{bcd}{a} = \frac{3}{a^2}.$$

Let $y = 3/x^2$, then $p(\sqrt{3/y}) = 0$ when $p(x) = 0$. Therefore rearranging $p(\sqrt{3/y}) = 0$ gives a polynomial of y with the required roots.

Senior Questions

1. (a) Firstly, we want to make the RHS of (1) look like the RHS of (2). Thus

$$j\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} = I_{motor}$$
$$Rj\frac{d^2\theta}{dt^2} + (Rc + k_m)\frac{d\theta}{dt} = R \times I_{motor} + k_m\frac{d\theta}{dt}$$

Hence

$$Rj\frac{d^2\theta}{dt^2} + (Rc + k_m)\frac{d\theta}{dt} = V_{in}. \qquad (3)$$

Let $\omega = \frac{d\theta}{dt}$. As $\omega \to \omega_\infty$, $\frac{d^2\theta}{dt^2} \to 0$. So

$$\begin{aligned}\omega_\infty &= \frac{V_{in}}{Rc + k_m} \\ &= \frac{12}{10(1) + 5} \\ &= 0.8 \text{ rads/s}\end{aligned}$$

(b) Similarly, if we set $\omega(0) = 0$, then

$$\begin{aligned}\frac{d^2\theta}{dt^2} &= \frac{V_{in}}{Rj} \\ &= \frac{12}{(10)(5)} \\ &= 0.24 \text{ rads/s}^2\end{aligned}$$

(c) If we wish to solve (3) for $\theta(t)$, we first re-write in terms of ω. Then

$$Rj\frac{d\omega}{dt} + (Rc + k_m)\omega = V_{in}$$
$$\frac{d\omega}{dt} + \frac{Rc + k_m}{Rj}\omega = \frac{V_{in}}{Rj}$$

Now the coefficient of ω is just a constant ($\frac{Rc+k_m}{Rj}$), and if we multiply both sides of the equation by $e^{t(Rc+k_m)/Rj}$, then the LHS turns into something that looks like the result of differentiation using the product rule. (This particular bit of magic is called the *method of integrating factors*.) Consequently,

$$\begin{aligned}
e^{t(Rc+k_m)/Rj}\frac{d\omega}{dt}+\frac{Rc+k_m}{Rj}e^{t(Rc+k_m)/Rj}\omega &= \frac{V_{in}}{Rj}e^{t(Rc+k_m)/Rj}\\
\frac{d}{dt}\left(\omega e^{t(Rc+k_m)/Rj}\right) &= \frac{V_{in}}{Rj}e^{t(Rc+k_m)/Rj}\\
\omega e^{t(Rc+k_m)/Rj} &= \frac{V_{in}}{Rj}\int e^{t(Rc+k_m)/Rj}\,dt\\
&= \frac{V_{in}}{Rj}\times\frac{Rj}{Rc+k_m}e^{t(Rc+k_m)/Rj}+C_1\\
&= \frac{V_{in}}{Rc+k_m}e^{t(Rc+k_m)/Rj}+C_1\\
\omega &= \frac{V_{in}}{Rc+k_m}+C_1e^{-t(Rc+k_m)/Rj}
\end{aligned}$$

Using the initial condition given in (b), we have $C_1=-\frac{V_{in}}{Rc+k_m}$, and hence

$$\omega(t)=\frac{V_{in}(1-e^{-t(Rc+k_m)/Rj})}{Rc+k_m}.$$

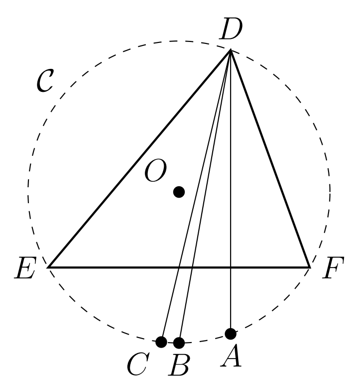
Since $\omega=\frac{d\theta}{dt}$,

$$\begin{aligned}
\theta(t) &= \frac{V_{in}}{Rc+k_m}\int(1-e^{-t(Rc+k_m)/Rj})\,dt\\
&= \frac{V_{in}}{Rc+k_m}\left(t+\frac{Rje^{-t(Rc+k_m)/Rj}}{Rc+k_m}+C_2\right)
\end{aligned}$$

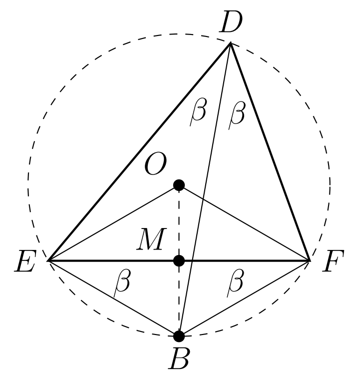
If we assume that $\theta(0)=0$, then $C_2=-\frac{Rj}{Rc+k_m}$, so

$$\theta(t)=\frac{V_{in}}{Rc+k_m}\left[t+\frac{Rj(e^{-t(Rc+k_m)/Rj}-1)}{Rc+k_m}\right].$$

2. Suppose that we already have a solution, $\triangle DEF$ with circumcircle, $\mathcal{C}$, centred at O and points A, B and C as shown below.



In particular, consider the angle bisector, BD. Let $\angle EDB = \beta$. Then $\angle EDB = \angle BDF = \beta$. Furthermore, since $\angle BDF$ and $\angle FEB$ subtend the same arc, $\angle FEB = \beta$. Similarly, it can be shown that $\angle BFE = \beta$, which implies that $\triangle BEF$ is isosceles with $EB = BF$. As OB and OF are both radii of the circle, $\triangle OEF$ is also isoceles. This means that $EOFB$ is a kite with diagonals EF and OB that intersect perpendicularly at M. Now since O is the centre of the circumcircle, OM is the perpendicular bisector of EF.



There are two consequences of this fact:

- M is the midpoint of EF, and hence the median DC passes through it.
- The altitude DA is parallel to OM.

So, if we are given the circumcircle and hte point A, B and C, we can construct the original triangle thus:

(a) Find O the centre of the circle.

(b) Draw a line between O and B.

(c) Draw a line parallel to OB passing through the point A. The second point of intersection between this line and the circle is the vertex D.

(d) Join the line DC. The point of intersection between DC and OB is M, the midpoint of the side EF.

(e) Draw a line through M perpendicular to AD. This line crosses the circle at E and F.

Some notes on the construction:

- Should the circumcentre lie outside the triangle, $EOFB$ will be a non-convex kite and OB must be extended to find M. The rest of the construction remains the same.
- There is one case where the above construction does not work: if the points A, B and C coincide, then at best we can say that $\triangle DEF$ is isosceles and determine the location of the vertex D, but the base EF cannot be determined.