Polynomial inequalities such as

\[
    x^2 - 7x > 18
\]

can be solved by collecting all terms on the left hand side to give

\[
    x^2 - 7x - 18 > 0,
\]

and then factorising:

\[
    (x - 9)(x + 2) > 0.
\]

The graph of \( y = (x - 9)(x + 2) \) is shown; we need to consider the case when \( y > 0 \), that is, the part of the graph above the \( x \)-axis, and find the corresponding \( x \) values. These \( x \)-values are given by \( x < -2 \) or \( x > 9 \), (*)&

the solution of the inequality. \textbf{Note.} Do not write this solution as \( "x < -2, x > 9" \) or as \( "9 < x < -2" \). Each of these means \( "x < -2 \) and \( x > 9" \), which is not the same as (*)&), and is \textbf{wrong}.

Inequalities where the unknown also appears in the denominator can be approached by multiplying out denominators \textbf{carefully} to obtain a polynomial inequality. For example, consider

\[
    x - 3 - \frac{4}{x} \leq 0.
\]

First we must note that \( x \) cannot be 0, because of the third term. Now to clear the denominator we do not multiply by \( x \): since \( x \) is unknown, we might be multiplying by a positive or a negative number, and we could not know whether we need to reverse the direction of the inequality. Instead, multiply both sides by \( x^2 \), which we know is positive. This gives \( x^3 - 3x^2 - 4x \leq 0 \), and we can factorise the left hand side to obtain the inequality

\[
    x(x + 1)(x - 4) \leq 0.
\]

We can now graph \( y = x(x + 1)(x - 4) \) and complete the solution as above. Alternatively, we may work with a number line, marking on it the points at which \( y = 0 \), that is, \( x = -1, 0, 4 \). These three points divide the real line into four intervals and we consider each separately.

- If \( x < -1 \) then \( x \) and \( x + 1 \) and \( x - 4 \) are all negative; so their product is negative; so this interval is part of our solution.
- If \( -1 < x < 0 \) then \( x + 1 \) is positive but \( x \) and \( x - 4 \) are negative; so the product is positive; so this interval is not part of our solution.
- The intervals \( 0 < x < 4 \) and \( x > 4 \) are treated in the same way.

Since the inequality is \( \leq \) rather than \( < \), the endpoints \(-1, 0, 4\) are included in the solution, except that we have already noted that \( x = 0 \) is impossible and must be excluded. Thus the solution can be illustrated as shown on the number line

\[
    x \leq -1 \quad \text{or} \quad 0 < x \leq 4.
\]
EXERCISES.

Please try to complete the following exercises. Remember that you cannot expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Solve the following polynomial inequalities both by sketching a graph and by using a sign diagram.
   (a) \((x - 3)(x - 7) \geq 0\);
   (b) \((x - 3)(x - 7) < 0\);
   (c) \((x + 4)(x - 2)(x - 9) > 0\);
   (d) \(x^2 - 7x + 6 \leq 0\);
   (e) \(x^2 + 4x < 12\);
   (f) \(x^3 - 13x + 12 \leq 0\);
   (g) \(x^3 - x^2 - 5x - 3 < 0\).

2. Solve the following by whichever method you find easiest.
   (a) \(\frac{6}{x + 1} \leq x + 2\);
   (b) \(2x + 5 - \frac{1}{x + 3} < 0\);
   (c) \(\frac{x - 2}{x - 2} \leq \frac{3}{x + 4}\);
   (d) \(\frac{x + 2}{x - 3} > 2\);
   (e) \(x - 2 + \frac{x}{x - 6} \geq 0\).

ANSWERS.

1. (a) \(x \leq 3\) or \(x \geq 7\);
   (b) \(3 < x < 7\); can be written \(x > 3\) and \(x < 7\);
   (c) \(-4 < x < 2\) or \(x > 9\);
   (d) \(1 \leq x \leq 6\);
   (e) \(-6 < x < 2\);
   (f) \(x \leq -4\) or \(1 \leq x \leq 3\);
   (g) \(x < -1\) or \(-1 < x < 3\); alternatively, \(x < 3\) and \(x \neq -1\).

2. (a) \(-4 \leq x < -1\) or \(x \geq 1\);
   (b) \(x < -\frac{7}{2}\) or \(-3 < x < -2\);
   (c) \(-4 < x < 2\) or \(x \geq 5\);
   (d) \(3 < x < 8\);
   (e) \(3 \leq x \leq 4\) or \(x > 6\).