The main algebraic property of surds is that if $x, y \geq 0$ then
\[ \sqrt{xy} = \sqrt{x} \sqrt{y} . \]

For example, we can simplify
\[ \sqrt{108} = \sqrt{36 \times 3} = \sqrt{36\sqrt{3}} = 6\sqrt{3} . \]

If you didn’t notice the best way to factorise 108, you could always have done the calculation one step at a time:
\[ \sqrt{108} = \sqrt{9 \times 12} = \sqrt{9\sqrt{12}} = 3\sqrt{12} \]
\[ = 3\sqrt{4 \times 3} = 3\sqrt{4\sqrt{3}} = 3 \times 2\sqrt{3} = 6\sqrt{3} . \]

Another example, this time involving division:
\[ \frac{15}{\sqrt{5}} = \frac{15}{\sqrt{5}} \sqrt{5} = \frac{15\sqrt{5}}{5} = 3\sqrt{5} . \]

Note that similar formulae do not hold for addition: in general,
\[ \sqrt{x + y} \neq \sqrt{x} + \sqrt{y} . \]

In particular, $\sqrt{x^2 + y^2}$ is not equal to $x + y$: equating these two expressions is a very common mistake! Occasionally we can do something like
\[ \sqrt{18} + \sqrt{50} = \sqrt{9 \times 2} + \sqrt{25 \times 2} = 3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2} , \]

but in most cases there is no useful simplification of $\sqrt{x} + \sqrt{y}$. We can, however, use basic algebra to multiply out sums and differences involving the same surd, for example,
\[ (3 - \sqrt{5}) (1 + 2\sqrt{5}) = 3 + 6\sqrt{5} - \sqrt{5} - 2 \times 5 = -7 + 5\sqrt{5} . \]

We can simplify certain expressions involving surds by rationalising the denominator. First notice that using the “difference of two squares” formula we have
\[ (a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c . \]

We can then do calculations like
\[ \frac{4 - \sqrt{7}}{1 + 2\sqrt{7}} = \frac{4 - \sqrt{7}}{1 + 2\sqrt{7}} \frac{1 - 2\sqrt{7}}{1 - 2\sqrt{7}} \]
\[ = \frac{18 - 9\sqrt{7}}{-27} \]
\[ = \frac{-2 + \sqrt{7}}{3} \]

and
\[ \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3} . \]

You do not always need to remove surds from the denominator. For example, we can write
\[ \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} , \]

but the right hand side is not really any simpler than the left.
EXERCISES.

Please try to complete the following exercises. Remember that you cannot expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop-in Centre.

1. Simplify the following:
   \[ \sqrt{28}, \sqrt{45}, \sqrt{48}, \sqrt{1100}, \sqrt{180}, \sqrt{567}. \]

2. Combine into a multiple of a single surd, as in the example at the bottom of page 1:
   \[ \sqrt{27} + \sqrt{300}, \sqrt{98} - \sqrt{8}, \sqrt{63 + 4\sqrt{175}}, \sqrt{245} + \sqrt{500}, \sqrt{10} + \sqrt{40} + \sqrt{90}. \]

3. Simplify:
   \[ \frac{1 + 2\sqrt{2}}{3 - \sqrt{2}}, \frac{1 + 2\sqrt{5}}{3 - \sqrt{5}}, \frac{7 + 2\sqrt{6}}{5 + 2\sqrt{6}}, \frac{5 + \sqrt{11}}{7 - 2\sqrt{11}}. \]

4. Simplify the following (the first step is given as a hint):
   \[ \frac{2\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} = \frac{2\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}, \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} = ? \]

   Then use similar ideas to simplify
   \[ \frac{2\sqrt{5} - 5\sqrt{2}}{\sqrt{5} + \sqrt{2}} \text{ and } \frac{3\sqrt{7} + 4\sqrt{6}}{\sqrt{7} - \sqrt{6}}. \]