MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 16, September 10, 2012

1. A box of apples costs \$4, a box of oranges costs \$3 and a box of lemons costs \$2. A person buys 8 boxes of fruit at a cost of \$23. If at least one box of each kind of fruit is bought, find the largest possible number of boxes of apples.

2. Suppose a, b are positive real numbers. Use the diagram below to give a **geometric proof** that $a^2 = (a-b)^2 + 2ab - b^2$.

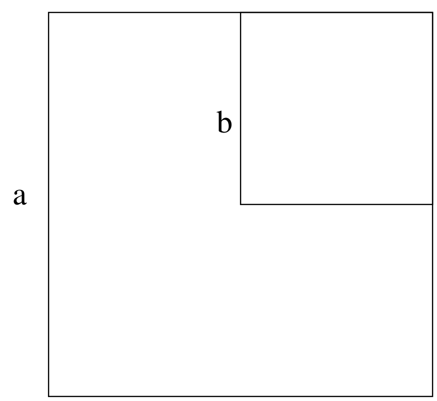


Figure 1: Two squares of length a and b

3. The perimeter of a rectangle is 20 cm what is the least value of the diagonal?

4. Consider the two sequences $x_0 = 1, x_1 = 1, x_{n+1} = x_n + 2x_{n-1}$ and $y_n = 8n + 1$. Prove that for $n > 1$ these two sequences never have a common term.

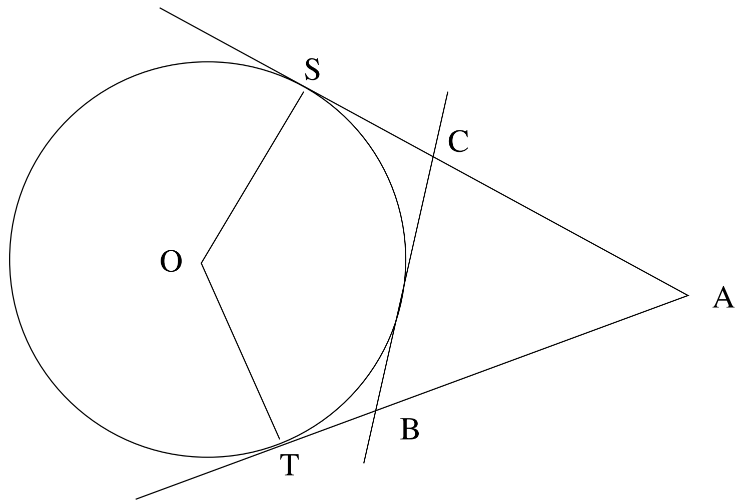
5. Use the fact that $a^2 + b^2 \geq 2ab$ for any positive real numbers a, b to show that, for a, b, c positive real numbers, $\frac{a^2+b^2+c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2$.

6. (a) Paul measured all 6 edges of a tetrahedron $ABCD$ and found them to be 1,3,4,5,6,8 cm. Can this be correct?

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

(b) Paul then measured the edges to be 2,3,4,5,6,8. If $AB = 2$ what is the length of CD?

7. A circle is drawn which touches BC in triangle ABC and also touches the two sides AB and AC produced at T and S respectively. Let O be the centre of this circle.



(a) Explain why OB bisects the angle TBC.

(b) Prove that the length of AT equals half the perimeter of the triangle ABC.

0.1 Senior Questions.

1. Prove that
$$\cos((n+2)\theta) = 2\cos((n+1)\theta)\cos\theta - \cos(n\theta),$$
for each integer $n \geq 0$.
Hence express $\cos 5\theta$ in terms of powers of $\cos\theta$.

2. For every positive real number $n > 1$, prove that
$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}.$$

3. Use the result in Q2 to prove that
$$2(\sqrt{N+1} - 1) < \sum_{n=1}^{N} \frac{1}{\sqrt{n}} < 2\sqrt{N}$$
and deduce that the sum of the first million terms of
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ...$$
is between 1998 and 2000.