

MATHEMATICS ENRICHMENT CLUB.¹

Problem Sheet 3, May 14, 2012

1. The perimeter of a base of a rectangular brick with integer sides is 18 cm, whilst its volume is 42 cm^3 . What is its height?

2. Calculate

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{2008}\right).$$

3. Find the smallest positive integer whose square ends in (a) 09 and (b) 9009.

4. Show that if a, b are positive numbers such that $ab \leq 1$ then

$$\frac{a}{b+1} + \frac{b}{a+1} + (1-a)(1-b) \leq 1.$$

5. Suppose we have the numbers $x_0 = 0, x_1 = 1$ and $x_{n+1} = x_n + 2x_{n-1}$ for $n \geq 2$.

a. Write down the numbers x_n for $n = 2, 3, 4, 5, 6$.

b. Show that there is no n for which $x_n = 1999$. (Hint: Use modulo 8 arithmetic).

c. Show that $x_n = \frac{2^n - (-1)^n}{3}$ satisfies the equation.

6. In $\triangle ABC$, extend the sides AB and AC and draw a circle outside the triangle which touches BC and these two produced sides. This circle is called the *escribed circle* of the triangle.

a. Show that $r_1 = \frac{A}{s-a}$, where r_1 is the radius of the escribed circle, A is the area of ABC , a is the length of BC and s is half the perimeter of ABC .

b. Show that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$, where r_2, r_3 are the radii of the other two escribed circles and r is the radius of the incircle (recall last weeks result.)

7. $ABCD$ is a parallelogram, Q a point inside it. Prove that the sum of the areas of AQB and CQD is half the area of $ABCD$.

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.

Senior Questions.

1. Find $\int_0^1 \frac{1}{1+t+t^2}$.

2. Find the limit $\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$.

3. A hand of eight cards is dealt from a standard pack. How many hands contain exactly three cards of the same value and the remaining cards from the remaining suit?