MATHEMATICS ENRICHMENT CLUB.\(^1\)
Problem Sheet 5, May 28, 2012

1. Two classes of 20 and 30 students average 66% and 56% respectively on an examination. What is the average for all the students on the exam?

2. A mathematics test has 5 questions on each of which people can score 0,1,2 or 3 marks. How many ways can a student receive a total of 12 marks for the test?

3. Mark the hours on a clockface with centre $O$ with the letters $A_1, A_2, \ldots, A_{12}$.
   
   (a) Find all the angles $XYO$, where $X$ and $Y$ are any hours.
   
   (b) What is the ratio of the areas of the quadrilaterals $A_{12}A_2A_6A_8$ and $A_{12}A_3A_6A_9$?

4. Find infinitely many integers $x$ such that
   
   $\sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}}$

   is an integer.

5. (a) Prove that $a + b \geq 2\sqrt{ab}$ for any positive real numbers $a, b$.
   
   (b) Deduce that for $x, y, z$ positive, $(x + y)(x + z)(y + z) \geq 8xyz$.

6. In the triangle $ABC$, it is given that $\angle ABC = 140^\circ$. Let $D$ be a point on $AC$ and $E$ a point on $AB$ such that the three triangles $AED, EDB$ and $DBC$ are all isosceles, with their vertices at $E, D$ and $B$ respectively. Find all the angles of the triangle $ABC$.

7. Let $ABCD$ be a trapezium and with $AB \parallel CD$. Let $M, N$ be the midpoints of $AD$ and $BC$ respectively. Show that $MN = \frac{1}{2}(AB + CD)$.

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\(^1\)Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
Senior Questions.

1. Let \( f(x) = \left(1 + \frac{1}{x}\right)^x \).

   (a) Prove that \( \frac{f''(x)}{f(x)} = \log \left(1 + \frac{1}{x}\right) - \frac{1}{1 + x} \).

   (b) By considering the area under the curve \( y = \frac{1}{t} \) for \( t \) from 1 to \( 1 + \frac{1}{x} \), show that
   \[ \log \left(1 + \frac{1}{x}\right) > \frac{1}{1 + x} \]
   and deduce that \( f(x) \) is increasing.

2. Suppose \( a > b > 0 \). Find \( \lim_{n \to \infty} \left(a^n + b^n\right)^{\frac{1}{n}} \).

3. By considering \( \cos(A + B) + \sin(A - B) = 0 \) find the general solution (for \( \theta \)) of
   \( \cos n\theta + \sin m\theta = 0 \).