MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 6, June 4, 2012

1. A parallelogram $ABCD$ has $BC = 4$ cm and $CD = 8$ cm. The point $A$ is 3 cm above $CD$. Find the length of the perpendicular from $A$ to $BC$.

2. If $a, b, c$ are real numbers and $a > b$, which of the following must be true?

   (a) $\frac{1}{a} > \frac{1}{b}$
   (b) $ac > bc$
   (c) $a^2 > b^2$
   (d) $a + c > b + c$
   (e) $\frac{1}{a} < \frac{1}{b}$.

3. (a) Verify that $x = 170, y = 39$ satisfy $x^2 = 19y^2 + 1$.
   (b) Hence find integers $x$ and $y$ such that $x^2 = 171y^2 + 1$ and $x^2 = 3211y^2 + 1$.

4. A rectangle has perimeter 20cm. What is the least value of the diagonal?

5. From the point $(x, y)$ we can move a counter to any one of the following points:

   $(2x, y), (x, 2y)$

   or

   $(x - y, y)$ if $x > y$, 
   $(x, y - x)$ if $y > x$.

   Starting from $(1, 1)$ can you see a rule to determine which points in the plane can be reached using the rules above?

6. The line joining a vertex of a triangle to the midpoint of the opposite side is called a median. Let $m_A$ denote the median in triangle $ABC$ from $A$ to $BC$.

   (a) Show that $AB + AC > 2m_A$. (Hint: Think about parallelograms)
   (b) Deduce that $AB + AC + BC > m_A + m_B + m_C$.

7. Given a circle $K$ with centre $O$ and diameter $AB$, let $C$ be any point on $K$.

   (a) Prove that $\angle ACB = 90^\circ$.
   (b) Describe how to construct a right-angled triangle $ACB$ if we are given its hypotenuse $AB$ and the length of the perpendicular dropped from $C$ to $AB$.

---

1 Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
Senior Questions.

1. Let \( S(x) = \frac{e^x - e^{-x}}{2} \) and \( C(x) = \frac{e^x + e^{-x}}{2} \).

   (a) Show that \( (C(x))^2 - (S(x))^2 = 1 \).

   (b) If \( S(x) = \tan \theta \), express \( C(x) \) in terms of \( \theta \).

2. Find the integral

\[
\int_{\pi/4}^{\pi/2} \frac{\cos^4 \theta}{\sin^2 \theta} \, d\theta.
\]

3. A die is thrown \( n \) times. Show that if the probability that a 6 appears at least once is greater than \( \frac{1}{2} \), then \( n > \frac{\log 2}{\log 6 - \log 5} \).