MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 15, September 3, 2013

1. Find all the positive integers such that $ab = 3a + 3b$.

2. Let K be the circumcircle through the vertices of a rectangle with sides a and b. On each side of the rectangle construct a semicircle. This will give four crescents formed between these semicircles and K. What is the sum of the areas of the four crescents?

3. Suppose that two non-parallel straight lines k and ℓ meet at a point P which is **not** on the page of my book. Construct a line which would (if P did lie on the page) bisect the angle between the lines and pass through P.

4. Suppose the last digit of $x^2 + xy + y^2$ is zero, and x and y are positive integers. Prove that the last **two** digits of $x^2 + xy + y^2$ are both zero.

5. (a) Use your calculator to show that $(2!)^{\frac{1}{2}} < (3!)^{\frac{1}{3}} < (4!)^{\frac{1}{4}}$.
   (b) Prove that for every integer $n > 0$, $(n!)^{\frac{1}{n}} < ((n+1)!)^{\frac{1}{n+1}}$.

6. (a) There are 128 coins of two different weights, 64 of each. How can one always find two different coins by performing no more than 7 weighings on a regular balance?
   (b) There are eight coins of two different weights, four of each. How can one always find two different coins by performing two weighings on a regular balance?

Senior Questions

1. The function $f(x) = x^x$ has an inverse $g(x)$ provided we restrict the domain of f to $x \geq 1$. Find a formula for the derivative of $g(x)$ in terms of x and $g(x)$.

2. The exponential function is defined as being the solution, $y(x)$, to the differential equations
$$y(x) = \frac{dy}{dx}, \quad \text{such that } y(0) = 1.$$
Prove $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.