UNSW THE UNIVERSITY OF NEW SOUTH WALES

Never Stand Still | Faculty of Science | School of Mathematics and Statistics

MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 3, May 21, 2013

1. The perimeter of a base of a rectangular brick with integer sides is 18 cm, whilst its volume is 42 cm^3. What is its height?

2. Calculate
$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{2008}\right).$$

3. Find the smallest positive integer whose square ends in (a) 09 and (b) 9009.

4. Show that if a, b are positive numbers such that $ab \leq 1$ then
$$\frac{a}{b+1}+\frac{b}{a+1}+(1-a)(1-b) \leq 1.$$

5. Suppose we have the numbers $x_0 = 0, x_1 = 1$ and $x_{n+1} = x_n + 2x_{n-1}$ for $n \geq 2$.
   (a) Write down the numbers x_n for $n = 2, 3, 4, 5, 6$.
   (b) Show that there is no n for which $x_n = 1999$. (Hint: Use modulo 8 arithmetic).
   (c) Show that $x_n = \frac{2^n-(-1)^n}{3}$ satisfies the equation.

6. In ΔABC, extend the sides AB and AC and draw a circle outside the triangle which touches BC and these two produced sides. This circle is called the *escribed circle* of the triangle.
   (a) Show that $r_1 = \frac{A}{s-a}$, where r_1 is the radius of the escribed circle, A is the area of ABC, a is the length of BC and s is half the perimeter of ABC.
   (b) Show that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$, where r_2, r_3 are the radii of the other two escribed circles and r is the radius of the incircle (recall last weeks result.)

7. $ABCD$ is a parallelogram, Q a point inside it. Prove that the sum of the areas of AQB and CQD is half the area of $ABCD$.

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Senior Questions

1. Prove that the square of the nth triangle number is the sum of the first n cubes, i.e.

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^3, \qquad \text{for } n \geq 1.$$

2. Find the limit $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+3^2+...+n^2}{n^3}$.

3. A hand of eight cards is dealt from a standard pack. How many hands contain exactly three cards of the same value and the remaining cards from the remaining suit?