MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 6, June 11, 2013

1. The product of the ages in years of two adults is 770. What is the sum of their ages?

2. An automatic card shuffler always re-arranges the cards in the same way. The cards begin in the order A,2,3,4,5,6,7,8,9,10,J,Q,K and after 2 shuffles the order is 6,5,K,10,Q,8,2,3,7,J,9,A,4. What order do we get if we shuffle them three times?

3. (a) Show that the median to the hypotenuse of a right-angled triangle has length exactly half the length of the hypotenuse.

   (b) Let A, B, C be a triangle with A_1, B_1, C_1 the midpoints of the sides BC, CA, AB respectively. Let D be the foot of the perpendicular from A to BC. Show that B_1C_1D is congruent to $B_1C_1A_1$.

4. Find all positive integers m and n such that $3m - 1$ is a multiple of n and $3n - 1$ is a multiple of m.

   (Hint: Suppose $m \leq n$, then n divides $3m - 1 < 3m \leq 3n$.)

5. We write $\phi(n)$ to be the number of positive integers less or equal to n and relatively prime to n (i.e. the number of numbers which have no common factor with n except 1.)

   (a) Find $\phi(12), \phi(30)$.

   (b) Suppose p is prime, find $\phi(p), \phi(p^2), \phi(p^3)$.

   (c) If p and q are two different primes, find $\phi(pq)$ (in factored form).

6. Suppose S is the intersection of the three medians in triangle ABC. A straight line is drawn through S parallel to BC meeting AC at T. What is the ratio of the area of AST to the area of ABC?

Senior Questions

1. Suppose that n is an odd integer greater than 3. Find the number of positive and negative (real) roots of $2x^n - nx^2 + 1 = 0$.

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

2. Let $f(x) = \left(1 + \frac{1}{x}\right)^x$.

(a) Prove that $\dfrac{f'(x)}{f(x)} = \log\left(1 + \dfrac{1}{x}\right) - \dfrac{1}{1+x}$.

(b) By considering the area under the curve $y = \frac{1}{t}$ for t from 1 to $1 + \frac{1}{x}$, show that $\log\left(1 + \dfrac{1}{x}\right) > \dfrac{1}{1+x}$ and deduce that $f(x)$ is increasing.