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MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 1, May 7, 2013

1. $\$2083\frac{1}{3}$ profit.

2. $n = 60 = 2^2 \times 3 \times 5$, which means the number of divisors is $(2+1)(1+1)(1+1) = 12$. This is the only one, given a prime factorisation $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ the number of divisors is $(m_1+1)(m_2+1)\cdots(m_k+1)$. For n to be divisible by $1, 2, 3, 4, 5, 6$ it must include in the prime factorisation $2^2 \times 3 \times 5$ at the very least. Adding exponenents or extra primes will only increase the number of divisors.

3. $$31^{2}4 < 32^{2}4 = 2^{5^{2}}4 = 2^{1}20 = 2^{8^{1}}5 = 256^{1}5 < 257^{1}5$$

4. (a) Conjecture that $S_n = 2^n(2n-1)(2n-3)\cdots(5)(3)(1)$ - that is, 2^n times all the odd numbers fomr $2n-1$ down to 1. Since every factor aside from 2^n is odd, the power of the 2 in the prime factorisation is n.

   (b) First prove $S_n = 2(2n-1)S_{n-1}$. Use this to prove our conjecture.

5. Letting $a = \sqrt[3]{5\sqrt{13}+18}$ and $b = \sqrt[3]{5\sqrt{13}-18}$, $x = a - b$ then we'll find that, after expanding $(a-b)^3$
$$x^3 = 36 - 3x,$$
the solution of which is $x = 3$.

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

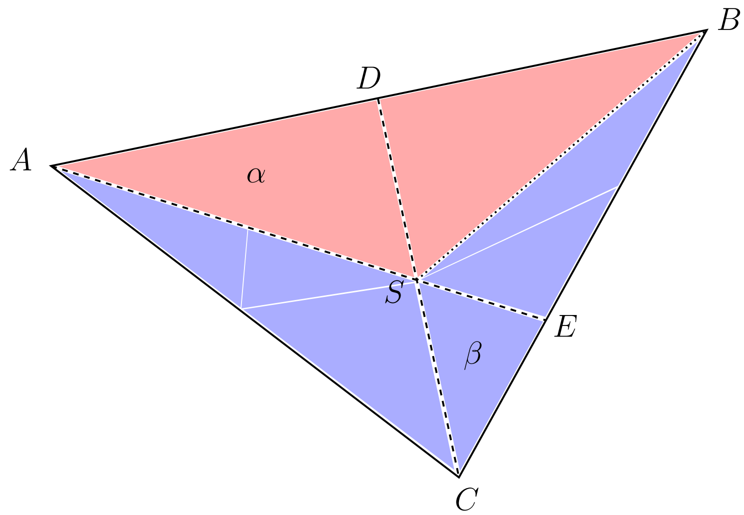

Figure 1: Showing the relevant areas for Question 6

6. Construct the line SB. Label the areas of ΔADS and ΔSEC, α and β respectively. Knowing that a cut from the baseline of a triangle to the opposite vertex divides the triangle into two triangles whose areas are in the same ratio as the two baselines, we can label each smaller triangle's area in terms of α and β as in Figure 6 (shown coloured so that each red block $= \alpha$ and each blue block $= \beta$). It can then be shown (with ΔAEC and ΔAEB that $\alpha = 3\beta$. Thus ΔADS and ΔASC are of equal area and so $|DS| : |SC| = 1 : 1$, and the areas of ΔASB and ΔBSE are in the ratio $6 : 2$ so $|AS| : |SE| = 3 : 1$.

7. (a) Consider an equilaterial triangle with side lengths x. Since P is interior the longest AP, BP or CP can be is x, but because ABP, ACP and BCP are triangles the sums of $|AP|$ and $|BP|$, $|AP|$ and $|CP|$ or $|BP|$ and $|CP|$ must each be larger than x and thus also the remaining length.

   (b) Take an isoceles triangle of height 3 and base length 1, and let P be close to the base. The long length is close to 3 while the sum of two shorter lengths will have a sum close to 1.