MATHEMATICS ENRICHMENT CLUB.\textsuperscript{1}  
Solution Sheet 1, May 7, 2013

1. $2083\frac{1}{3}$ profit.

2. $n = 60 = 2^2 \times 3 \times 5$, which means the number of divisors is $(2 + 1)(1 + 1)(1 + 1) = 12$. This is the only one, given a prime factorisation $n = p_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}$ the number of divisors is $(m_1 + 1)(m_2 + 1) \cdots (m_k + 1)$. For $n$ to be divisible by $1, 2, 3, 4, 5, 6$ it must include in the prime factorisation $2^2 \times 3 \times 5$ at the very least. Adding exponents or extra primes will only increase the number of divisors.

3. 

$$31^2 < 32^2 < 32^4 = 2^5 < 4 = 2^{10} = 2^{11} < 5^2 = 2^5 15 = 25615 < 257^15$$

4. (a) Conjecture that $S_n = 2^n(2n - 1)(2n - 3) \cdots (5)(3)(1)$ - that is, $2^n$ times all the odd numbers from $2n - 1$ down to $1$. Since every factor aside from $2^n$ is odd, the power of the $2$ in the prime factorisation is $n$.

(b) First prove $S_n = 2(2n - 1)S_{n-1}$. Use this to prove our conjecture.

5. Letting $a = \sqrt[3]{5\sqrt{13} + 18}$ and $b = \sqrt[3]{5\sqrt{13} - 18}$, $x = a - b$ then we’ll find that, after expanding $(a - b)^3$

$$x^3 = 36 - 3x,$$

the solution of which is $x = 3$.

\textsuperscript{1}Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
6. Construct the line $SB$. Label the areas of $\triangle ADS$ and $\triangle SEC$, $\alpha$ and $\beta$ respectively. Knowing that a cut from the baseline of a triangle to the opposite vertex divides the triangle into two triangles whose areas are in the same ratio as the two baselines, we can label each smaller triangle’s area in terms of $\alpha$ and $\beta$ as in Figure 6 (shown coloured so that each red block = $\alpha$ and each blue block = $\beta$). It can then be shown (with $\triangle AEC$ and $\triangle AEB$ that $\alpha = 3\beta$. Thus $\triangle ADS$ and $\triangle ASC$ are of equal area and so $|DS| : |SC| = 1 : 1$, and the areas of $\triangle ASB$ and $\triangle BSE$ are in the ratio $6 : 2$ so $|AS| : |SE| = 3 : 1$.

7. (a) Consider an equilateral triangle with side lengths $x$. Since $P$ is interior the longest $AP$, $BP$ or $CP$ can be is $x$, but because $ABP$, $ACP$ and $BCP$ are triangles the sums of $|AP|$ and $|BP|$, $|AP|$ and $|CP|$ or $|BP|$ and $|CP|$ must each be larger than $x$ and thus also the remaining length.

(b) Take an isosceles triangle of height 3 and base length 1, and let $P$ be close to the base. The long length is close to 3 while the sum of two shorter lengths will have a sum close to 1.