MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 11, August 6, 2013

1. (a)

$$\begin{aligned} 0 &\le (a-b)^2 \quad \text{with equality only if } a=b \\ 0 &\le a^2+b^2-2ab \\ ab &\le \frac{a^2+b^2}{2} \end{aligned}$$

so ab is largest when $a=b$, and since $a+b=k$ then at $a=b=\frac{k}{2}$.

(b) From above, first note that $xy \le \frac{c^2}{2}$, then

$$\begin{aligned} c^4 &= (x^2+y^2)^2 \\ c^4 &= x^4+y^4+2x^2y^2 \\ x^4+y^4 &= c^4-2x^2y^2, \end{aligned}$$

which is minimum when x^2y^2 is maximum, which from above is when $x=y$ and has a value of $\left(\frac{c^2}{2}\right)^2$. So the minimum value of $x^4+y^4=c^4-\frac{c^4}{2}=\frac{c^4}{2}$.

2. Construct the triangles APB and AQB. Let P' be at the intersection of the circle and the line AP, now since AB is a diameter and P' on the circle, triangle $AP'B$ is right at P', which also means triangle $PP'B$ is right at P', and so $\angle APB = \angle P'PB < 90°$.

   Similarly extend AQ to the circle and call the intersection Q', then $\angle AQ'B$ is right, which implies $\angle Q'QB < 90°$. Since $\angle Q'QB$ is an external angle of triangle AQB we have $\angle Q'QB = \angle QAB + \angle QBA$, and so $\angle QAB + \angle QBA < 90°$ and so $\angle AQB > 90°$.

3. (a) Suppose a quadratic is factorised with roots α and β, then it is

$$(x-\alpha)(x-\beta) = x^2-(\alpha+\beta)x+\alpha\beta.$$

Since 2 343 643 is odd, one of α or β must be odd and the other even, but the product of an odd and even number is even, and so cannot be 2 382 987. Hence there are no integer solutions.

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

(b) By similar logic, both α and β must be even (to have even sum *and* even product). If they are both even, then the product $\alpha\beta$ must be divisible by 4 (write $\alpha = 2m$ $\beta = 2n$ then $\alpha\beta = 4nm$), but 2 382 982 is not, and hence there are no integer solutions.

4. To make \$10 out of n 50c coins and m 20c coins we must satisfy

$$5n + 2m = 100, \quad n, m \in \mathbb{Z},\ n, m > 0$$

or

$$m = 100 - 5\frac{n}{2}, \quad m, n \in \mathbb{Z},\ n, m > 0.$$

So we merely count the number of n which are divisible by 2 and satisfy the above, of which there are 9.

5. (a) In general, if we prime factorise $x = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ then every factor can be written as $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where each $a_= 0, 1, \ldots, m_i$. So there are $m_1 + 1$ choices for a_1, $m_2 + 1$ choices for a_2 and so on, and hence the number of factors is $(m_1 + 1)(m_2 + 2) \cdots (m_k + 1)$. Then $20 = 2^2 \times 5$ and so has $3 \times 2 = 6$ factors, so $\tau(20) = 6$.
If $n = p_1^{m_1} \cdots p_k^{m_k}$, then $n^2 = p_1^{2m_1} \cdots p_k^{2m_k}$ and so $\tau(n^2) = (2m_1 + 1)(2m_2 + 1) \cdots (2m_k + 1)$ which is a product of odd numbers and hence odd and so cannot be equal to the even number $2\tau(n)$.

(b) The number $144^2 = (3 \times 2^2)^4 = 3^4 \times 2^8$ so $\tau(144^2) = 5 \times 9 = 45$ and $\tau(144) = \tau((3 \times 2^2)^2) = \tau(3^2 \times 2^4) = 3 \times 5 = 15$ and $3 \times 15 = 45$.

6. Let $\sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}} = a + b = y$, and recall

$$y^3 = (a + b)^3 = a^3 + b^3 + 3ab(a + b).$$

Now

$$\begin{aligned} a^3 + b^3 &= 2x \\ ab &= \sqrt[3]{x^2 - (x^2 + 1)} \\ &= \sqrt[3]{1} = 1. \end{aligned}$$

so

$$\begin{aligned} y^3 &= 2x + 3 \times 1 \times y \\ x &= \frac{3y - y^3}{2}, \qquad \text{where } y \in \mathbb{Z}. \end{aligned}$$

Senior Questions

1. Expand the right hand side:

$$\begin{aligned} (n^2 - (3n + 1))^2 - 25n^2 &= n^4 - 2n^2(3n + 1) + (3n + 1)^2 - 25n^2 \\ &= n^4 - 6n^3 - 2n^2 + 9n^2 + 6n + 1 - 25n^2 \\ &= n^4 - 6n^3 - 18n^2 + 6n + 1. \end{aligned}$$

Now if $n^4 - 6n^3 - 18n^2 + 6n + 1$ is prime its only factors are itself and 1. Since

$$n^4 - 6n^3 - 18n^2 + 6n + 1 = (n^2 - 3n - 1 - 5n)(n^2 - 3n - 1 + 5n) = (n^2 - 8n - 1)(n^2 + 2n - 1)$$

we must have one of the latter factors equal to 1. So first consider

$$n^2 - 8n - 2 = 0$$

but $(-8)^2 - 4 \times 1 \times (-2) = 72$ which is not a square, so this one has no integer solutions. Also

$$n^2 + 2n - 2 = 0$$

has no integer solutions. So there are no integer n for which $n^4 - 6n^3 - 18n^2 + 6n + 1$ is prime.

2. It doesn't matter which order the players win their games in, just the total number of games won or lost. If one player wins n games, the probability they won them all by luck alone is

$$\binom{100}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{100-n} = \frac{1}{2^{100}} \binom{100}{n}.$$

So we must find N such that $\sum_{n=1}^{N} \frac{1}{2^{100}} \binom{100}{n} = 0.05$. This is prohibitively hard, but we can approximate the binomial distribution with a normal distribution with mean 50 and variance 25 (or standard deviation of 5). In a normal distribution 95% of values lie within 2 standard deviations of the mean, which means our players must win more than 60 of the 100 games to demonstrate that it is unlikely they won by luck alone.