1. Start with

\[
\frac{x + 3y}{2x + 5y} = \frac{4}{7} \\
7(x + 3y) = 4(2x + 5y)
\]

\[
y = x
\]

but we mustn’t have \(2x + 5y = 0\), which means \(x \neq -\frac{5}{2}y\). The final solution then is \(x = y \neq 0\).

2. Solving simultaneously we get \(a = 0\), \(b = 6\), \(c = 7\), \(d = 3\) and \(e = -1\), so \(c\) is the largest. Can you do this without solving simultaneously?

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Figure 1: Solution for Question 3

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1 Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni. Solution to question 4 provided by G. Liang
4. Begin by constructing the equilateral triangle $ADB$. Draw the line $CD$ to intersect $AB$ at $E$. Draw $EG$ parallel to $DB$ and $EF$ parallel to $DA$. Connect $F$ and $G$ then triangle $EFG$ is equilateral. To prove this is true, construct $B'A'$ parallel to $BA$ and passing through $D$, and show that $BB'D$ and $AA'D$ are similar to $GBE$ and $FAE$ respectively.

![Figure 2: Solution for Question 4](image)

5. Suppose otherwise, i.e., there is a $50 \times 50$ corner with no queens on it. Let’s suppose the empty corner is the bottom-left (the argument will hold if any corner is chosen). To fit 100 queens on the chess board, all not attacking each other, we must at least have 1 queen per row, and 1 queen per column. Since the bottom-left corner is empty, the 50 top rows must have a queen each, and the 50 left columns must also have a queen each. Since none are attacking each other, there can be no queens in the top-right (otherwise not all rows/columns would have exactly one queen). Now we count the number of diagonals that stretch from the top-left $50 \times 50$ corner to the bottom-right $50 \times 50$ corner; there are 99, and hence not enough diagonals for one per queen, and so two queens must share a diagonal. This is a contradiction, so we cannot have an empty corner.

6. (a) Just expand the two right-hand sides:

\[
(x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1) = x^{15} - x^{12} + x^{12} - x^9 + x^9 - x^6 + x^6 - x^3 + x^3 - 1
= x^{15} - 1
\]

\[
(x^5 - 1)(x^{10} + x^5 + 1) = x^{15} - x^{10} + x^{10} - x^5 + x^5 - 1
= x^{15} - 1.
\]

(b) From part (a) $2^{15} - 1$ has factors $2^5 - 1 = 31$ and $2^3 - 1 = 7$, both of which are prime. We can use long division to divide $2^{10} + x^5 + 1$ by $2^3 - 1$, which gives
us the remainder of dividing $2^{15} - 1$ by both $2^5 - 1$ and $2^3 - 1$. So the prime factorisation is $2^{15} - 1 = 31 \times 7 \times 151$.

(c) We can use

\[ x^{15} + 1 = (x^3 + 1)(x^{12} - x^9 + x^6 - x^3 + 1) = (x^5 + 1)(x^{10} - x^5 + 1). \]

So we can write

\[ 2^{15} + 1 = (2^5 + 1)(2^3 + 1)R = 33 \times 9 \times R. \]

Using long division we get $R = 138 + \frac{1}{3}$ so

\[ 2^{15} + 1 = 3^3 \times 11 \times \left( \frac{3 \times 138 + 1}{3} \right) = 3^2 \times 11 \times (3 \times 110 + 1) = 3^2 \times 11 \times 331. \]