MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 15, September 3, 2013

1. Rearranging we get

$$3a = b(a-3)$$
$$b = \frac{3a}{a-3}.$$

So $3a$ has to be divisible by $a-3$, for example, a can be 4. But then if $a > 4$ and odd, $a-3$ will be even, and $3a$ will be odd, so $a-3$ does not divide $3a$, so a must be even. Let $a = 3n$, then we have

$$b = \frac{3n}{n-1}$$

so $n = 2$ and then $a = 6$.

These are the only solutions, $a = 4, b = 12$ and $a = 6, b = 6$.

2. The radius of the circumcircles is $\frac{1}{2}\sqrt{a^2+b^2}$. The sum of the area of the 4 crescents, is the sum of the 4 semi circles, plus the area of the rectangle, then with the area of the circumcircle subtracted. So

$$\begin{aligned} A &= 2\frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + 2\frac{1}{2}\pi\left(\frac{b}{2}\right)^2 + ab - \pi\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 \\ &= ab + \pi\left(\frac{a^2}{4} + \frac{b^2}{4} - \frac{a^2}{4} - \frac{b^2}{4}\right) \\ &= ab. \end{aligned}$$

3. Refer to figure 1. Lines AP and BP are k and ℓ. First construct the angle bisector of $\angle PAB$ and $\angle PBA$. These angle bisectors meet at O, and must also meet the angle bisector of $\angle APB$ at O. Draw $A'B'$ parallel to AB and through O, and repeat, forming the angle bisectors of $\angle PA'B'$ and $\angle PB'A'$. These meet at O' and because the triangles PAB and $PA'B'$ are similar, O' also lies on the angle bisector of $\angle APB$. Connect O and O' and you've drawn the angle bisector of $\angle APB$.

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni. Some problems from Tournament of Towns

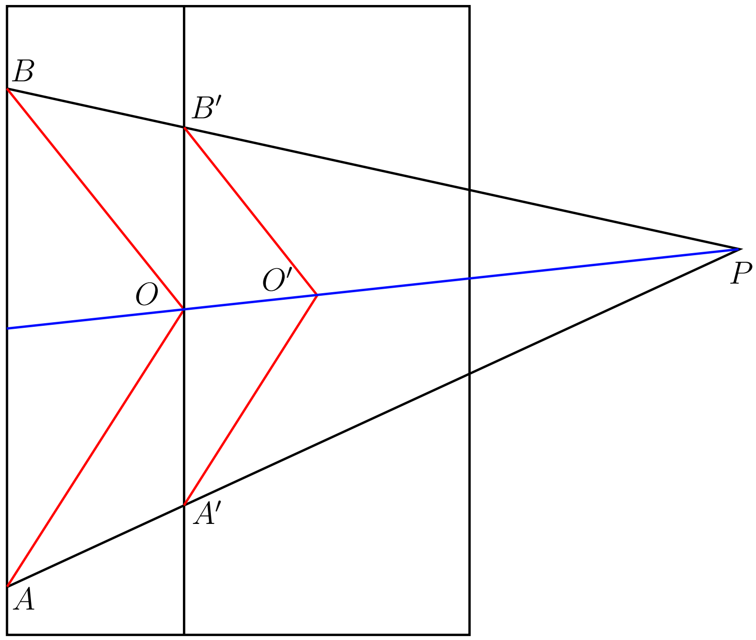

Figure 1: Construction for question 3

4. If either x or y is odd, $x^2 + xy + y^2$ is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, $x^2 + xy + y^2$ is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one of x^2 or y^2 ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6 and so $x^2 + xy + y^2$ is not a multiple of 10. So the only possibility is that x and y are both multiples of 10, meaning $x^2 + xy + y^2$ is a multiple of 100.

5. (a) N/A

   (b) Starting with $n! = n(n-1)(n-2)\cdots 1$, we know that

$$\begin{aligned}
n! &< \underbrace{nnn\cdots n}_{n \text{ times}} \\
&< n^n \\
&< (n+1)^n \\
(n!)^{\frac{1}{n}} &< n+1 \\
(n!)(n!)^{\frac{1}{n}} &< (n+1)n! \\
(n!)^{1+\frac{1}{n}} &< (n+1)! \\
(n!)^{\frac{1+n}{n}} &< (n+1)! \\
(n!)^{\frac{1}{n}} &< ((n+1)!)^{\frac{1}{n+1}}.
\end{aligned}$$

6. (a) Let the two types of coins be A and B. Split the 128 coins into two piles of 64 each. Now we weigh the two piles. If they are the same weight, this means that both piles have the same number of A coins, so we discard one of the piles, and

split the remainder into two equal piles. If we have the good fortune that our two piles are always of equal weight, then after 6 weighings we have 2 coins left, and they must be of different type.

Suppose now that at some point the two piles are not of equal weight. Now take half the coins from each pile and weigh them. If they are equal in weight, then discard these coins and continue with the others. If they are different in weight, discard the others and continue with these coins.

We will show that there is always at least one of each type remaining. Suppose at any step, n type A coins remain. We show that it is impossible to remove all n coins. If the two new piles are even in weight, then we can only remove $\frac{n}{2}$. If they are uneven in weight then we can remove at most all but 1 type A coin (if we happen to only select 1 A coin for the second weighing).

(b) Weigh 4 of the coins against the other 4. If they balance, discard one set. Weigh 2 of theremaining coins against the other 2. If they balance, take both coins from one side. If not, take 1 coin from each side.

Suppose one side is heavier in the first weighing. Weigh 2 of these coins against the other 2. If they balance, all 4 are heavy. Take 1 of them and 1 from the lighter side in the first weighing. If they do not balance, then the heavier side consists of 2 heavy coins and the lighter side consists of 1 heavy and 1 light coin, so take the two coins on the lighter side.