MATHEMATICS ENRICHMENT CLUB.\footnote{Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.}

Solution Sheet 2, May 14, 2013

1. \( LCM(10, 12) = 2^2 \times 3 \times 5 = 60 \) minutes.

2. 6528(10\(a\) + 3) = 8256(30 + \(a\)) implies \(a = 4\)

3. 3 play only the piano.

4. Let \( p(x) = (3 + 2x + x^2)^{1998} = a_0 + a_1x + a_2x^2 + \cdots \).
   
   (a) \( a_0 = p(0) = 3^{1998}\), \(a_1 = p'(0) = 1998(3 + 2 \times 0 + 0^2)^{1997} \times (2 + 2 \times 0) = 1998 \times 3^{1997} \times 2\)
   
   (b) \( a_0 + a_1 + a_2 + \cdots = p(1) = (3 + 2 + 1)^{1998} = 6^{1998}\)
   
   (c) \( a_0 - a_1 + a_2 - \cdots = p(-1) = (3 - 2 + 1)^{1998} = 2^{1998}\).

5. (a) Since \( a + b + c = 2 \) and \( a + b > c \), \( a + c > b \) and \( b + c > a \) each \( a, b, c < 1\).
   
   (b)
   
   \[
   (1 - a)(1 - b)(1 - c) > 0
   \]

   \[
   1 - (a + b + c) + ab + bc + ca - abc > 0
   \]

   \[
   -1 + ab + bc + ca - abc > 0
   \]

   \[
   (a + b + c)^2 = 4
   \]

   \[
   a^2 + b^2 + c^2 + 2(ab + bc + ca) = 4
   \]

   \[
   ab + bc + ca = 2 - \frac{1}{2}(a^2 + b^2 + c^2).
   \]

   Combining the two yields the answer.

6. (a) Using the triangle inequality gives \( AC < AB + BC \), \( AC < AD + DC \), \( BD < AD + AB \) and \( BD < BC + CD \), summing all of these together gives

   \[
   2(AC + BD) < 2(AB + BC + CD + AD)
   \]

   \[
   AC + BD < p.
   \]
Mark as $E$ the intersection of $AC$ and $BD$, then again using the triangle inequality we have $AB < AE + EB$, $BC < EB + EC$, $CD < CE + ED$ and $AD < ED + AE$. Again summing all of these together gives

$$AB + BC + CD + AD < AE + EC + BE + ED + BE + EB + ED + CE + EA$$
$$p < AC + BD + BE + CA$$
$$p < 2(AC + BD)$$
$$\frac{1}{2}p < AC + BD.$$ 

(b) The lines $AE$, $BE$, $CE$ and $DE$ divide the quadrilateral into 4 pieces. Say $\angle AEB = \theta$, and $\angle BEC = \phi$, then by opposite angles $\angle CED = \theta$ and $\angle AED = \phi$. The 4 angles must sum to $2\pi$ so $2\theta + 2\phi = 2\pi \implies \phi = \pi - \theta$. Note also that $\sin \theta = \sin(\pi - \theta)$. Now we may sum the area of the 4 triangles to determine the area of the quadrilateral:

$$a = \frac{1}{2}AE \cdot BE \sin \theta + \frac{1}{2}BE \cdot CE \sin \phi + \frac{1}{2}CE \cdot DE \sin \theta + \frac{1}{2}DE \cdot AE \sin \phi$$
$$= \frac{1}{2} \sin \theta (AE \cdot BE + BE \cdot CE + CE \cdot DE + DE \cdot AE)$$
$$= \frac{1}{2} \sin \theta (AE + EC)(BE + ED)$$
$$= \frac{1}{2} \sin \theta AC \cdot BD \leq \frac{1}{2}AC \cdot BD.$$ 

(c) We can see from the above that equality of the last expression holds if $\sin \theta = 1 \implies \theta = \frac{\pi}{2}$, thus if $AC \cdot BD = 2a$ then the diagonals are perpendicular.
7. From the diagram we can see three sets of rhombi, each with side length $r$, the radius of the three circles, which are $O_2BO_3P$, $O_3CO_1P$ and $O_1AO_2P$ (drawn in black). Each is a rhombus because all 4 sides are equal. Thus $O_2B||O_3P||O_1C$ and so $O_1O_2BC$ is a parallelogram (a pair of equal length, parallel sides) and hence $O_1O_2$ is equal in length to $CB$. Similarly $O_1O_3BA$ and $O_2O_3CA$ are parallelograms so $O_1O_2$ and $BA$ are equal in length and $O_2O_3$ and $CA$ are equal in length. Thus $O_1O_2O_3$ is congruent to $ABC$ since three sides are equal.