MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 2, May 14, 2013

1. $LCM(10, 12) = 2^2 \times 3 \times 5 = 60$ minutes.

2. $6528(10a + 3) = 8256(30 + a)$ implies $a = 4$

3. 3 play only the piano.

4. Let $p(x) = (3 + 2x + x^2)^{1998} = a_0 + a_1 x + a_2 x^2 + \cdots$.

   (a) $a_0 = p(0) = 3^{1998}$, $a_1 = p'(0) = 1998(3+2\times 0+0^2)^{1997}\times(2+2\times 0) = 1998\times 3^{1997}\times 2$

   (b) $a_0 + a_1 + a_2 + \cdots = p(1) = (3 + 2 + 1)^{1998} = 6^{1998}$

   (c) $a_0 - a_1 + a_2 - \cdots = p(-1) = (3 - 2 + 1)^{1998} = 2^{1998}$.

5. (a) Since $a + b + c = 2$ and $a + b > c$, $a + c > b$ and $b + c > a$ each $a, b, c < 1$.

   (b)

$$\begin{aligned} (1-a)(1-b)(1-c) &> 0 \\ 1 - (a+b+c) + ab + bc + ca - abc &> 0 \\ -1 + ab + bc + ca - abc &> 0 \end{aligned}$$

   and

$$\begin{aligned} (a+b+c)^2 &= 4 \\ a^2 + b^2 + c^2 + 2(ab + bc + ca) &= 4 \\ ab + bc + ca &= 2 - \frac{1}{2}(a^2 + b^2 + c^2). \end{aligned}$$

   Combining the two yields the answer.

6. (a) Using the triangle inequality gives $AC < AB + BC$, $AC < AD + DC$, $BD < AD + AB$ and $BD < BC + CD$, summing all of these together gives

$$\begin{aligned} 2(AC + BD) &< 2(AB + BC + CD + AD) \\ AC + BD &< p. \end{aligned}$$

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Mark as E the intersection of AC and BD, then again using the triangle inequality we have $AB < AE+EB$, $BC < EB+EC$, $CD < CE+ED$ and $AD < ED+AE$. Again summing all of these together gives

$$\begin{aligned} AB + BC + CD + AD &< AE + EC + BE + ED + BE + ED + CE + EA \\ p &< AC + BD + BE + CA \\ p &< 2(AC + BD) \\ \frac{1}{2}p &< AC + BD. \end{aligned}$$

(b) The lines AE, BE, CE and DE divide the quadrilateral into 4 pieces. Say $\angle AEB = \theta$, and $\angle BEC = \phi$, then by opposite angles $\angle CED = \theta$ and $\angle AED = \phi$. The 4 angles must sum to 2π so $2\theta + 2\phi = 2\pi \implies \phi = \pi - \theta$. Note also that $\sin\theta = \sin(\pi - \theta)$. Now we may sum the area of the 4 triangles to determine the area of the quadrilateral:

$$\begin{aligned} a &= \frac{1}{2}AE \cdot BE \sin\theta + \frac{1}{2}BE \cdot CE \sin\phi + \frac{1}{2}CE \cdot DE \sin\theta + \frac{1}{2}DE \cdot AE \sin\phi \\ &= \frac{1}{2}\sin\theta \left(AE \cdot BE + BE \cdot CE + CE \cdot DE + DE \cdot AE\right) \\ &= \frac{1}{2}\sin\theta \left(AE + EC\right)\left(BE + ED\right) \\ &= \frac{1}{2}\sin\theta AC \cdot BD \leq \frac{1}{2}AC \cdot BD. \end{aligned}$$

(c) We can see from the above that equality of the last expression holds if $\sin\theta = 1 \implies \theta = \frac{\pi}{2}$, thus if $AC \cdot BD = 2a$ then the diagonals are perpendicular.

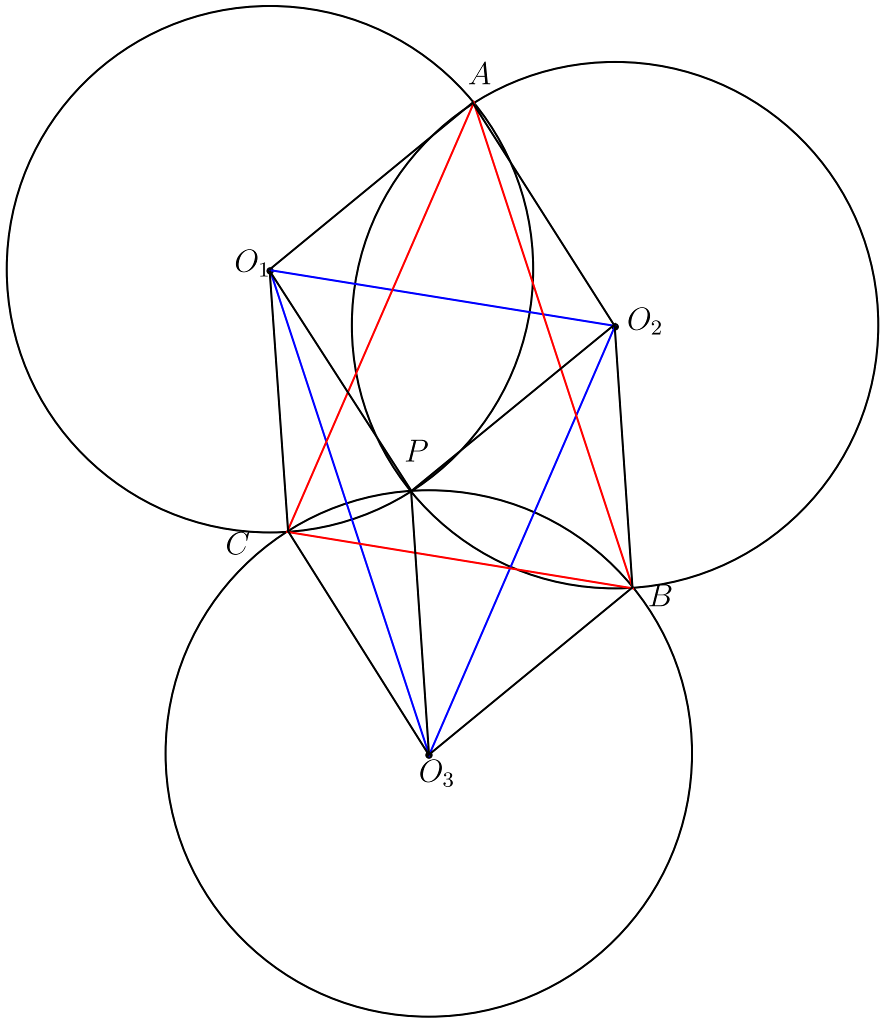

Figure 1: Diagram for question 7

7. From the diagram we can see three sets of rhombi, each with side length r, the radius of the three circles, which are O_2BO_3P, O_3CO_1P and O_1AO_2P (drawn in black). Each is a rhombus because all 4 sides are equal. Thus $O_2B||O_3P||O_1C$ and so O_1O_2BC is a parallelogram (a pair of equal length, parallel sides) and hence O_1O_2 is equal in length to CB. Similarly O_1O_3BA and O_2O_3CA are parallelograms so O_1O_2 and BA are equal in length and O_2O_3 and CA are equal in length. Thus $O_1O_2O_3$ is congruent to ABC since three sides are equal.