MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 3, May 21, 2013

1. The dimensions of the brick are integers L, W, H with $L + W = 9\ cm$ and $LWH = 42\ cm^3 \implies H = 42/(L+W)\ cm$. Only $L = 2$, $W = 7$ has LW divide 42, and so $H = 3\ cm$.

2.
$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{2008}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2007}{2008}\right)$$
$$= \frac{1}{2008}.$$

3. (a) Let $n = b_l 10^l + b_{l-1}10^{l-1} + \cdots b_1 10 + b_0$ and $n^2 = a_k 10^k + \cdots a_1 10 + a_0$, with $a_0 = 9$, $a_1 = 0$. By squaring n we see that $a_0 = b_0^2 \mod 10$ and $a_1 = \left\lfloor \frac{b_0^2}{10} \right\rfloor + 2b_1 b_0 \mod 10$. Thus $b_0 = 3$ and $0 = 6b_1 \mod 10$. The smallest $b_1 > 0$ which satisfies this is $b_1 = 5$ so $n = 53$.

   (b) This time $a_0 = a_3 = 9$, $a_1 = a_2 = 0$. If we write $p(x) = b_l x^l + \cdots b_1 x + b_0$, then let $p(x)^2 = c_j x^j + \cdots + c_1 x + c_0$ then

$$\begin{aligned}
c_0 &= b_0^2\\
c_1 &= 2b_0 b_1\\
c_2 &= 2b_0 b_2 + b_1^2\\
c_3 &= 2b_0 b_3 + 2b_1 b_2\\
&\vdots
\end{aligned}$$

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

$$\begin{aligned}
a_0 &= c_0 \mod 10\\
a_1 &= \left(\left\lfloor \frac{c_0}{10} \right\rfloor + c_1\right) \mod 10\\
a_2 &= \left(\left\lfloor \frac{c_1 + \left\lfloor \frac{c_0}{10} \right\rfloor}{10} \right\rfloor + c_2\right) \mod 10\\
a_3 &= \left(\left\lfloor \frac{c_2 + \left\lfloor \frac{c_1 + \left\lfloor \frac{c_0}{10} \right\rfloor}{10} \right\rfloor}{10} \right\rfloor + c_3\right) \mod 10
\end{aligned}$$

Solving in order from b_0 to b_3 one finds $b_0 = 3$, $b_1 = 5$ or 0. Then if $b_1 = 5$ we find no solution for b_3, so $b_1 = 0$. Then $b_2 = 0$ or 5, but this time if $b_2 = 0$ we find no solution for b_3, thus $b_2 = 5$ and we find $b_3 = 1$. That is, $n = 1503$.

4.

$$\frac{a}{b+1} + \frac{b}{a+1} + (1-a)(1-b) = \frac{1+a+b+a^2b^2}{1+a+b+ab}.$$

Since $ab < 1$, $a^2b^2 < ab$, and the result follows.

5. (a)

$$\begin{aligned}
x_2 &= 1\\
x_3 &= 3\\
x_4 &= 5\\
x_5 &= 11\\
x_6 &= 21.
\end{aligned}$$

(b) Note that 1999 in base 8 is 3717. Note also that in base 8 x_3, x_4, x_5 and x_6 end in either 3 or 5. By writing $x_n = a_k 8^k + a_{k-1}8^{k-1} + \cdots + 3$ and $x_{n-1} = b_l 8^l + b_{l-1}8^{l-1} + \cdots + 5$ deduce that x_{n+1}, in base 8, ends with a 3. Generalise to deduce that all x_n, $n \geq 3$ end in either 5 or 3 and hence can never equal 3717.

(c) Validate by substituting into the recursive rule $x_{n+1} = x_n + 2x_{n-1}$.

6. (a) Let O be the centre of the escribed circle, P, Q and R the points at which it touches AB, AC and BC respectively. Using properties of isoceles triangles deduce that $|AP| = |AQ| = s$, then divide the quadrilateral $APOQ$ into the triangles ABC, BPR, CQR and the quadrilateral $PRQO$ and compare areas.

(b) Note that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{A}$. By dividing the triangle ABC up into 3 pairs of congruent, right triangles using the radii of the incircle we can deduce that $A = rs$ where r is the radius of the incircle. The result then follows.

7. Since $ABCD$ is a parallelogram, there's a line through Q perpendicular to both AB and CD. Let P and R be the points this line connects to AB and CD respectively, then let $|PQ| = h_1$, $|QR| = h_2$, and $|AB| = |CD| = b$. Then the sum of the areas of ABQ and CDQ is $\frac{1}{2}h_1 b + \frac{1}{2}h_2 b = \frac{1}{2}b(h_1 + h_2)$ which is half of the area of a parallelogram with base b and perpendicular height $h_1 + h_2$, which is what $ABCD$ is.

Senior Questions

1. Solve using induction, or visit `http://en.wikipedia.org/wiki/Squared_triangular_number#Proofs` for a cute geometrical representation.

2. $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n)$, so $\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{3}$.

3. (I could be wrong) Choose one of 13 values for the triplet and one of 4 suits to exclude and there are 13×4 possible triplets, then $\binom{12}{5}$ combinations of the remaining suit are left. So there are $13 \times 4 \times \binom{12}{5}$ possible hands.