MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 6, June 11, 2013

1. The prime factorisation of $770 = 2 \times 5 \times 7 \times 11$, so assuming by adults we mean over 18 year olds, our two people are 22 and 35.

2. (**Disclaimer:** Introduction 'group theory' answer - this question can be answered more simply by deductive logic, or guess and check (maximum 13 guesses), but this question's close ties to group theory I think warrants a bit of abstract algebra. If you just want the answer, skip to the end ☺)

   Let's write the card shuffler as a function σ, where $\sigma(n)$ is the new position of the nth card after one shuffle. We'll also write iterated shuffles as σ^m, meaning m compositions of the shuffling function σ. As a final piece of notation, we'll introduce 'k-cycles', which are written as a collection of numbers in a pair of brackets and indicate that the σ value of each number is that to its immediate right (or the first position if at the end of the cycle), e.g. $(1\ 2\ 3)$ means $1 \to 2$, $2 \to 3$ and $3 \to 1$.

   The information given tells us

   $$\sigma^2 = (1\ 12\ 5\ 2\ 7\ 9\ 11\ 10\ 4\ 13\ 3\ 8\ 6).$$

   We can multiply (compose) cycles together just by tracing from left (i.e. applying the cycles to each number left to right), for example

   $$(1\ 2\ 3)(2\ 1\ 4) = (1)(2\ 3\ 4) = (2\ 3\ 4)$$

   since $1 \to 2 \to 1$, $2 \to 3$, $3 \to 1 \to 4$ and $4 \to 2$.[2] In this manner we can repeatedly multiply σ^2 and we find

   $$\sigma^{26} = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13),$$

   i.e. shuffling 26 times puts the cards back in to the order they originally were. This means the 'order' of σ is ≤ 26, where the 'order' of a permutation is how many times

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

[2] An interesting result is that every permutation can be written as a product of 2-cycles, e.g. $(1\ 2\ 3) = (1\ 3)(3\ 2)$, and even though this 2-cycle representation is not unique, it is always made up of either an odd or even number of 2-cycles.

you multiply it by itself to get the identify function - one that leaves everything alone like the one above.

Since σ is, at most, a 13-cycle its order is ≤ 13. So the order of σ could be $1, 2, 13$ or 26 in order to satisfy $\sigma^{26} = ()$, but it can't be 26, it's not 1 or 2 from the given information, so it must have order 13.

So now we work out σ^{12}, then we can determine σ so that $\sigma^{12}\sigma = ()$. I worked out σ^{12} by first performing

$$\sigma^2\sigma^2 = \sigma^2 = (1\ 5\ 7\ 11\ 4\ 3\ 6\ 12\ 2\ 9\ 10\ 13\ 8)$$

then

$$\sigma^8 = \sigma^4\sigma^4 = (1\ 7\ 4\ 6\ 2\ 10\ 8\ 5\ 11\ 3\ 12\ 9\ 13)$$

and finally

$$\sigma^{12} = \sigma^8\sigma^4 = (1\ 11\ 6\ 9\ 8\ 7\ 3\ 2\ 13\ 5\ 4\ 12\ 10).$$

To find σ I then wrote it as a 2-cycle representation

$$\sigma = (a\ 1)(b\ 2)(c\ 3)(d\ 4)\cdots(m\ 13)$$

and work through, from left to right, making sure I put the numbers back where they started. For instance $\sigma^{12}(1) = 11$, so set $a = 11$, $\sigma^{12}(2) = 13$, so $b = 13$, $\sigma^{12}(3) = 2$ so $c = 13$ (I've already made $b = 13$, and so far $2 \to 13$ so now I make $13 \to 3$ after, so that overall $2 \to 13$). Continuing, we find

$$\begin{aligned}\sigma &= (11\ 1)(13\ 2)(13\ 3)(12\ 4)(12\ 5)(9\ 6)(13\ 7)(13\ 8)(13\ 9)(11\ 10)(11\ 12)(11\ 13)\\ &= (1\ 10\ 12\ 4\ 5\ 13\ 2\ 3\ 7\ 8\ 9\ 6\ 11).\end{aligned}$$

Finally, this means the cards originally ordered $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$ become, after one shuffle, $J, K, 2, Q, 4, 9, 3, 7, 8, A, 6, 10, 5$.

3. (a) Draw the right angled triangle ABC with right angle at C. Let D be the midpoint of AB, and E a point on AC such that $AC \perp DE$. Then ΔADE is similar to ΔABC (three angles equal). Since $AD = \frac{1}{2}AB$ then $AE = \frac{1}{2}AC$ or rather $AE = EC$. Now ΔAED is congruent to ΔCED (two sides equal, $AE = EC$, DE common, and an included angle $\angle AED = \angle DEC$). Thus $\frac{1}{2}AB = AD = DC$.

   (b) From part i) we see $DB_1 = B_1C$ and $DC_1 = C_1B$. Note that ΔCB_1A_1 is similar to ΔCAB (two sides in ratio and an included angle). The sides are in ratio $1 : 2$ so $A_1B_1 = \frac{1}{2}AB = C_1B$, and so $A_1B_1 = DC_1$. Similarly ΔBC_1A_1 is similar to ΔBAC, so $C_1A_1 = B_1C = B_1A_1$. Thus ΔB_1C_1D and $\Delta B_1C_1A_1$ are congruent because they have 3 equal sides.

4. Following the hint, we must have $3m - 1 = n$ or $3m - 1 = 2n$, since $3m - 1 < 3n$. So

$$\begin{aligned}3(3m-1) - 1 &= km, \quad k \in \mathbb{Z}\\ (9-k)m &= 4\\ m &= \frac{4}{9-k}\\ m &= 4, 2, \text{ or } 1,\end{aligned}$$

or

$$\begin{aligned}
3\frac{3m-1}{2}-1 &= km\\
9m-3-2 &= 2km\\
m &= \frac{5}{9-2k}\\
m &= 5,\ \text{or } 1.
\end{aligned}$$

Thus the pairs are $(1,1)$, $(1,2)$, $(2,5)$, $(4,11)$ and $(5,7)$.

5. (a) $\phi(12)=4$, $\phi(30)=8$

   (b) We can think of $\phi(n)$ as being the number of numbers less than n which are not a multiple of a factor of n (except the factor 1). So if p is prime, its only factors are 1 and p, so every other number is not a multiple of a factor that isn't 1, except p itself. Thus $\phi(p)=p-1$.

   For p^2, the factors are $1,p$ and p^2, so the multiples of the factors that aren't 1 are $p,2p,3p,\ldots,p^2$, of which there are p. So $\phi(p^2)=p^2-p$.

   For p^3, the factors are $1,p,p^2$ and p^3, so the multiples of the factors that aren't 1 are $p,2p,3p,\ldots,p^2,(p+1)p,\ldots,2p^2,(2p+1)p,\ldots$, that is, the multiples of p^2 are contained in the multiples of p, of which there are p^2. So $\phi(p^3)=p^3-p^2$.

   (c) Using the same method as above, the factors of pq are $1,p,q$ and pq, so the multiples of the factors that aren't 1 are $p,2p,3p,\ldots,qp$ (q of them) and $q,2q,3q,\ldots,pq$ (p of them), but we don't want to count pq twice. So $\phi(pq)=pq-q-(p-1)$.

6. We use the fact that the medians divide ABC into 2 equal area pieces, and that S is $\frac{2}{3}$ along the median from A (you can prove these by considering the areas of smaller triangles with the same heights).

   Let the median from A meet BC at P, since ST is parallel to BC triangles APC and AST are similar - 3 angles equal. Since $AS=\frac{2}{3}AP$ then the area of AST is $\frac{4}{9}$ the area of APC which is half the area of ABC so the area of AST is $\frac{2}{9}$ the area of ABC.

Senior Questions

1. Let $f(x)=2x^n-nx^2+1$, then $f'(x)=2nx(x^{n-2}-1)$. So f has stationary points at $x=0$ and $x=1$ (since $n>3$ and odd). Taking the second derivative $f''(x)=2n(n-1)x^{n-2}-2n$, so $f''(0)=-2n<0$ and $f''(1)=2n(n-1)-2n=2n(n-2)>0$. So $x=0$ is a local max and $x=1$ is a local min.

   Finally $f(0)=1>0$ and $f(1)=3-n<0$. Since these are the only stationary points, f is monotonic between/outside of them. Since $x=0$ is a local max, and positive there is one root for $x<0$, which is unique since f is monotonic decreasing for $x<0$. Since $f(0)>0>f(1)$ and f is monotonic between 0 and 1 there is exactly one root for $0<x<1$. Since $x=1$ is a local min, $f(1)<0$ and $f(x)$ is monotonic increasing for $x>1$ there is exactly one root for $x>1$. Thus, in total, there are 3 roots.

2. Take the log of both sides and the differentiate both sides with respect to x.

$$\begin{aligned} \log f(x) &= x \log\left(1+\frac{1}{x}\right) \\ \frac{f'(x)}{f(x)} &= \log(\left(1+\frac{1}{x}\right) - \frac{1}{x+1}. \end{aligned}$$

3. Draw the graph of $y = \frac{1}{t}$ for t between 1 and $1 + \frac{1}{x}$ and we see that the area under the curve is larger than the area of the rectangle with base $1 + \frac{1}{x} - 1$ and height $\frac{1}{1+\frac{1}{x}}$, so

$$\int_1^{1+\frac{1}{x}} \frac{1}{t}\, dt = \log\left(1+\frac{1}{x}\right) > \frac{1}{x}\frac{x}{x+1} = \frac{1}{1+x}.$$

Thus $\frac{f'(x)}{f(x)} > 0$, and since $f(x) > 0$ for all x so is $f'(x)$.