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MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 7, June 18, 2013

1. (a) Using the cosine rule on angle A we get the two equations

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and

$$h^2 = b^2 + \left(\frac{c}{2}\right)^2 - 2b\frac{c}{2}\cos A.$$

By eliminating $\cos A$ we arrive at the desired result.

(b) The medians of the triangle will divide it into 3 smaller triangles whose sides are, for example, $\frac{2}{3}h$, $\frac{2}{3}k$ and a, with one median $\frac{1}{3}l$. So we can use the formula from part (a) to determine the length a. Repeat for b and c and knowing the lengths of the three sides is sufficient to construct the triangle.

2. Draw the triangles AO_1P and PO_2B. Both of which are isoceles triangles as they have two sides which are radii. Also $\angle O_1PO_2 = \pi$ since both circles have a common tangent at P. So $\angle O_1AP = \angle O_1PA = \angle O_2PB = \angle O_2BP$, using the properties of isoceles triangles and the vertically opposite angle. Now since they are both tangents at A and B, they make an angle of $\frac{\pi}{2}$ with the radii O_1A and O_2B, meaning the angles between the tangents and AB are equal and alternate, implying the two tangets are parallel.

3. Start by noting that if 100 divides $n!$ then 100 divides $m!$ for $m \geq n$, and so $n! + (n+1)! + \cdots$ contributes nothing extra to the final two digits. So we want to find the smallest n so that 100 divides $n!$ then sum the final digits of $1! + 2! + \cdots + n!$, which we can do by ignoring all other digits.

Now, $100 = 2^2 5^2$ in prime factorisation form, so the prime factorisation of $n!$ must have two 2s and two 5s. Let's start with 9!:

$$\begin{aligned} 9! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \\ &= 2 \times 3 \times 2^2 \times 5 \times (2.3) \times 7 \times (2^3) \times (3^2) \\ &= 2^7 \times 3^4 \times 5 \times 7. \end{aligned}$$

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Finding the next, 10!, we simply multiply by the prime factorisation of $10 = 2 \times 5$ so

$$10! = 2^8 \times 3^4 \times 5^2 \times 7.$$

We can see that 10 is the smallest number n for which 100 divides $n!$. So let's add:

last two digits of $n!$	n
1	1
2	2
6	3
24	4
20	5
20	6
40	7
20	8
80	9

summing the left hand column gives 13.

For 3 digits, we do the same thing, but find the smallest n such that 1000 divides n (which is 15).

4. Take the plane through A, B_1 and D_1 and shift it $\frac{1}{2}$ a side length along AD (which also shifts it halfway along AB, or halfway up DD_1 and BB_1 or halfway along D_1C_1 or B_1C_1). Thus all those midpoints are coplanar. We can also see that each side length of the hexagon produced is $\sqrt{2}$ times the side length of the cube, and that each angle is equal (try rotating the cube about the line through AC).

5. Recall that the lengths of two tangents to an external point are equal. Using this we can write the perimeter of the quadrilateral as $2s = 2w + 2x + 2y + 2z$ where w, x, y, z are the lengths of the tangents to the vertices of the quadrilateral. Also recall that a radius and tangent are perpendicular, so the quadrilateral is split into 4 pairs of congruent, right triangles, so the area is $A = 2\left(\frac{1}{2}rw + \frac{1}{2}rx + \frac{1}{2}ry + \frac{1}{2}rz\right) = rs$.

6. Each number on a clock is $\frac{1}{12}360° = 30°$ apart, and each minute-notch is $\frac{1}{60}360° = 6°$ apart. But, since the hour hand also moves throughout the hour we must correct for that - the hour hand moves $\frac{1}{60}30° = \frac{1}{2}°$ every minute. So the angle between the two hands is $5 \times 30° - 25 \times \frac{1}{2}° = \frac{275}{2}° = 137.5°$.

Senior Questions

1. Use the double angle formula for tan to find a double angle formula for cot, and use this to take the cot of both sides. Use the same formula to get an exact expression for $\cot\frac{\pi}{12}$ $(\sqrt{3} = \cot\frac{\pi}{6} = \cot\left(2\frac{\pi}{12}\right))$.

2. Smallest is $\frac{20}{8} = \frac{5}{2}$ and largest is $\frac{40}{4} = 10$.