MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 9, July 23, 2013

1. Since the sequence is arithmetic $a_n = a_1 + (n-1)d$ where d is the common difference. With $a_1 = 10$ this means $a_n = 10 + (n-1)d$. Now using the data $a_{a_2} = 100$ we get

$$\begin{aligned} 100 &= 10 + (a_2 - 1)d \\ 90 &= (a_2 - 1)d \\ 90 &= (10 + (2-1)d)d \\ 0 &= d^2 + 10d - 90, \qquad \text{so} \\ d &= \frac{-9 \pm \sqrt{81 + 4 \times 90}}{2} \\ d &= -15, 6 \qquad \text{and since } d > 0,\ d = 6. \end{aligned}$$

Now we can compute $a_{a_{a_3}}$, so

$$\begin{aligned} a_{a_{a_3}} &= 10 + (a_{a_3} - 1)d \\ &= 10 + ((10 + (a_3 - 1)d) - 1)d \\ &= 10 + ((10 + ((10 + (3-1)d) - 1)d) - 1)d \\ &= 10 + ((10 + ((10 + 2 \times 6) - 1) \times 6) - 1) \times 6 \\ &= 10 + ((10 + (22 - 1) \times 6 - 1) \times 6 \\ &= 10 + (10 + 126 - 1) \times 6 \\ &= 10 + 135 \times 6 \\ &= 820. \end{aligned}$$

2. Let's talk about two moves, one makes a number bigger, and one reduces the number by one. First, take the latter, every $n = (n-1) + 1$ and $n - 1 = (n-1) \times 1$. To make a number bigger take instead $n = (n-2) + 2$ so $2(n-2) = 2n - 4$ which is larger than n for $n > 4$. Using these two moves we can make the moves

$$22 \to 40 \to 76 \to 148 \to 292 \to 580 \to 1156 \to 2308 \to 2307 \to 2306 \to \cdots \to 2001$$

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

3. Note

$$\begin{aligned}
125^{100} &= 5^{300} \\
&= \frac{10^{300}}{2^{300}} \\
&= \frac{10^{300}}{(1024)^{30}} \\
&= \frac{10^{300}}{(1.024 \times 10^3)^{30}} \\
&= \frac{10^{300}}{1.024^{30} \times 10^{90}} \\
&= \frac{10^{210}}{1.024^{30}}.
\end{aligned}$$

It now remains to see that $1.024^{30} < 10$ and hence there are 210 digits. Can you show this using the binomial theorem?

4. (a) See figure 1

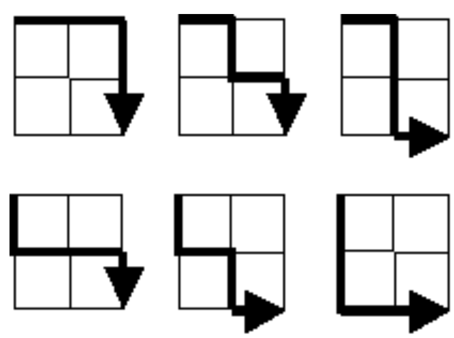

Figure 1: The possible paths on a 2×2 grid. .

(b) At each vertex ask how many paths are there from the top-left to the given vertex. In fact, it is the sum of the number of paths from the top-left to the vertex immediately to the left of the given vertex and the number of paths from the top-left to the vertex immediately above the given vertex. This means the number of paths from the top-left to each vertex follows a Pascal's Triangle pattern. So there are $\binom{40}{20}$ paths.

(c) This generalises easily to $\binom{2n}{n}$.

5. Let's label the vertices of our hexagon 1 through to 6. Then we can refer to edges as (xy) where x and y are a vertex. Now since we can just re-label the hexagon however we want, let's just consider vertex 1. There are 5 edges coming off vertex 1, and since we only have 2 colours, 3 of these edges are the same colour, let's say red. And again, since we can re-label the hexagon however we want, let's suppose these edges are (12),(13) and (14). Suppose we don't have any red triangles, so (23) must be blue to prevent $\Delta 123$ from being red, also (34) must be blue to prevent $\Delta 134$ from being

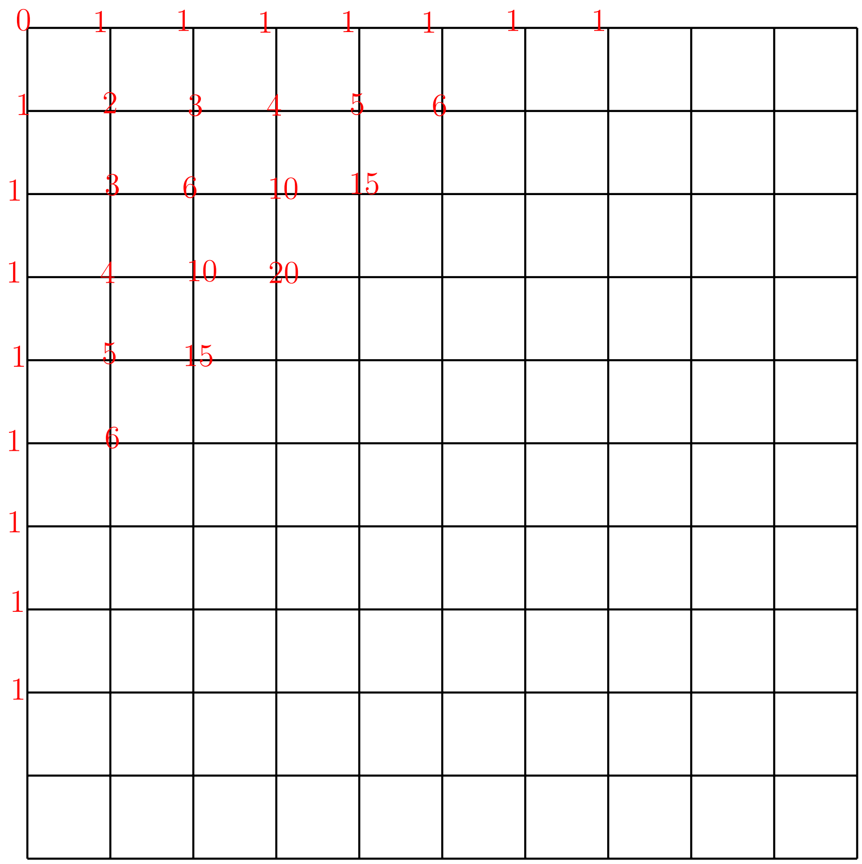

Figure 2: Pascals triangle emerging

all red and (24) must be blue to prevent $\Delta 124$ from being all red. But now (23), (34) and (24) are all blue, so $\Delta 234$ is blue.

Seeing as it doesn't matter how you label the hexagon, this always happens.

6. Look at the sequences in base-8, x_n always ends in either 3 or 5, while y_n always ends in 1. Hence they can never be the same.