UNSW AUSTRALIA

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MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 14, August 26, 2014[1]

1. Two tennis players, A and B are equally strong (have a 50/50 probability of beating the other). Is it more likely for A to beat B in 3 sets out of 4, or in 5 sets out of 8?

2. Scissors, paper, rock is a balanced game. Two players each choose one of the options, scissors, paper or rock, at the same time. The winner is determined using the rules *rock beats scissors*, *scissors beats paper* and *paper beats rock* – all other combinations are draws. By balanced we mean that each option has an equal chance of winning. Show that adding an extra option necessarily makes the game unbalanced – one option will win more often, or with higher probability, than the others.

3. (a) Let a and b be integers, and p an odd prime which divides both $a+b$ and a^2+b^2. Prove that p divides both a and b.

   (b) Let p be a prime greater than 3, and a, b and c be integers such that p divides $a+b+c$, $a^2+b^2+c^2$ and $a^3+b^3+c^3$. Prove that p divides each of a, b and c.

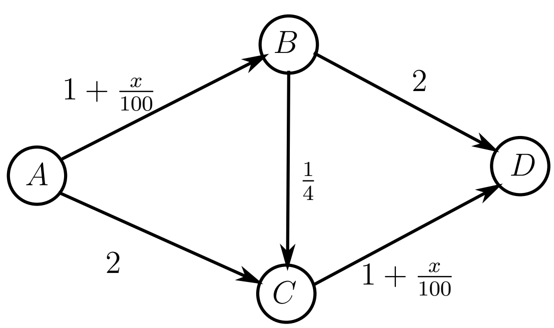

Figure 1: Picture for question 4

4. The picture shows a traffic network for cars travelling from A to D. Each road is labelled with the time it will take each car to travel it, where x is the percentage of cars driving on that road. For instance, if all cars travel ABD, it will take them $1+\frac{100}{100}+2=4$ hours, whereas if half the cars travel ABD and the other half travel $ABCD$ those on ABD take $1+\frac{100}{100}+2=4$ hours whilst those on $ABCD$ take $\left(1+\frac{100}{100}\right)+\frac{1}{4}+\left(1+\frac{50}{100}\right)=\frac{15}{4}$ hours.

[1]Some problems from UNSW's publication *Parabola*

(a) There are 3 possible routes – ABD, $ABCD$ and ACD. Find the number of cars that should take each route so that everyone has the same travel time. This assignment of number of cars to routes is called the *equilibrium*, and their travel time is called the *equilibrium travel time*.

(b) Show that you can remove one road to make the equilibrium travel time quicker.

5. Any fraction can be written as an egyptian fraction, which is the sum of the reciprocals of distinct integers. For example $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$. Note that $\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$ is not an egyptian fraction expansion as it uses $\frac{1}{3}$ twice. Let p and q be odd primes, write $\frac{2}{pq}$ as an egyptian fraction.

6. Prove that a number made up of 3^n equal digits is divisible by 3^n.

Senior Questions

1. A bottle is made by taking a sphere of radius R and cutting a hole of radius $r < R$ in the top. Water is poured into this bottle at the rate of a $\mathrm{L.s^{-1}}$. How long will it take for the bottle to overflow?

2. A napkin holder is made from a sphere of radius R by drilling a hole of height $h < 2R$ through its centre. Show that the volume of the napkin holder – the material that remains after the drilling – depends only on h.