MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 15, September 2, 2014[1]

1. Which is larger 100^{300} or $300!$?

2. A chess board is an 8×8 grid of squares coloured white or black so that no two adjacent squares are the same colour. Given tiles that are 2×1 grid squares it is possible to cover the chessboard completely, and it takes precisely 32 tiles. Show that it is impossible to cover a chessboard with opposite corners removed.

3. Find all 3 digit numbers which are equal to the sum of the factorials of their digits.

4. The triangle ΔABC is right with hypotenuse AB. Two semi-circles are drawn on the exterior of the triangle with diameters AC and BC. The semi-circle through ABC is also drawn. Compute the sum of the areas bounded by the two arcs which meet at A and C, and the two arcs which meet at B and C.

5. Tic-tac-toe is a game played by two players who take turns marking either x or o in a square on a 3×3 grid. A player wins if they get 3 of their symbols in a row, but if the grid is filled without a winner the game is a draw. How many tic-tac-toe games end in a draw?

6. (a) Let p denote the perimeter of a triangle and r the radius of its inscribed circle. Show that $r \leq \frac{p}{6\sqrt{3}}$.

   (b) $P_1, P_2, \ldots, P_n$ are the vertices of an n-gon inscribed in a circle. The centre, O, of the circle lies inside the n-gon. Let A denote the sum of the areas of the cirlces inscribed in the n triangles $\Delta P_1OP_2, \Delta P_2OP_3, \ldots, \Delta P_nOP_1$, and let B denote the area of the n-gon. Show that $\frac{A}{B} \leq \frac{\pi}{3\sqrt{3}}$. If $0 < k < \frac{\pi}{3\sqrt{3}}$ show that for any $n \geq 3$ there exists a polygon with $k = \frac{A}{B}$.

Senior Questions

1. Let f and g be real valued, continuous functions defined for $-1 \leq x \leq 1$. Denote by

$$\langle f, g\rangle = \left(\int_{-1}^{1} (f(x)g(x))^2 \, dx\right)^{\frac{1}{2}}.$$

[1]Some problems from UNSW's publication *Parabola*

Two functions, f and g, are said to be *orthogonal* if $\langle f, g\rangle = 0$. Let $p_0(x) = 1$ and $p_1(x) = x$,

(a) Show that p_0 and p_1 are orthogonal.

(b) Find $p_2(x)$, a degree 2 polynomial, which is orthogonal to both p_0 and p_1.

(c) Show that any polynomial of degree less than or equal to 2 can be written as a *linear combination* of p_0, p_1 and p_2, i.e. for any polynomial g with $\deg g \leq 2$, there exists constants α_0, α_1 and α_2 such that

$$g(x) = \alpha_0 p_0(x) + \alpha_1 p_1(x) + \alpha_2 p_2(x).$$